Section 16.2. Matchings in Bipartite Graphs

Note. In the previous section, we motivated the exploration of matchings in bipartite graphs with the Assignment Problem (Problem 16.2). In this section we give necessary and sufficient conditions for the existence of certain such matchings (and perfect matchings of bipartite graphs). We define coverings of a graph (in terms of a set of vertices) and relate maximum matchings and minimum coverings (in Theorem 16.7).

Theorem 16.4. Hall's Theorem.
A bipartite graph $G = G[X, Y]$ has a matching which covers every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$ (where $N(S)$ is the set of all vertices which are neighbors of some vertex in S).

Note. Hall's Theorem (Theorem 16.4; due to P. Hall in "On Representatives of Subsets," *Journal of the London Mathematical Society*, **10**, 26–30 (1935) and available online on the Dartmouth College website) is sometimes called the *Marriage Theorem.* As Bondy and Murty describe it in rather dated language (see page 426): "...if every group of girls in a village collectively like at least as many boys as there are girls in the group, then each girl can marry a boy she likes." As the title of Hall's paper suggests, the results can be stated in other terms.

Definition. Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a finite family of not necessarily sitinct) subsets of a finite set A. A *system of distinct representatives* (SDR) for family $\mathcal{A}$ is a set $\{a_i \mid i \in I\}$ of distinct elements of A such that $a_i \in A_i$ for all $i \in I$.

Note. Hall's Theorem (Theorem 16.4) can b expressed in terms of SDRs as follows. Finite family of sets $\mathcal{A}$ has a system of distinct representatives if and only if $|\cup_{i \in J} A_i| \geq |J|$ for all subsets J of indexing set I. To conclude this, let $G = G[X, Y]$ where $X = I$ and $Y = A$, where the edges are defined in terms of neighbors as $N(i) = A_i$ for all $i \in I$. Hall himself used this version of his result to prove a result in group theory concerning representatives of left and right cosets of a group (which is easy if the cosets are based on a normal subgroup, but not easy in general). See Exercise 16.2.21. In terms of perfect matchings of bipartite graphs, Hall's Theorem implies the following.

Corollary 16.5. A bipartite graph $G[X, Y]$ has a perfect matching if and only if $|X| = |Y|$ and $|N(S)| \geq |S|$ for all $S \subseteq X$.

Corollary 16.6. Every nonempty regular bipartite graph has a perfect matching.

Note. We have used the terms "cover/covering" in two different senses. In Section 2.4. Decompositions and Coverings we considered a family of subgraphs of graph G such that every edge of G is contained in at least one of the subgraphs. In Section 8.6, "Linear and Integer Programming," and 12.1 Stable Sets we considered a set of vertices that "cover" every edge of the graph, as in the following sense.

Definition. A *covering* of a graph G is a subset K of $V(G)$ such that every edge of G has at least one end in K. A covering K^* is a *minimum covering* if G has no covering K with $|K| < |K^*|$. The number of vertices in a minimum covering of G is the *covering number* of G, denoted $\beta(G)$. A covering is a *minima; covering* if none of its proper subsets is itself a covering.

Note. Minimal and minimum coverings of the Petersen graph are given in Figure 16.7.

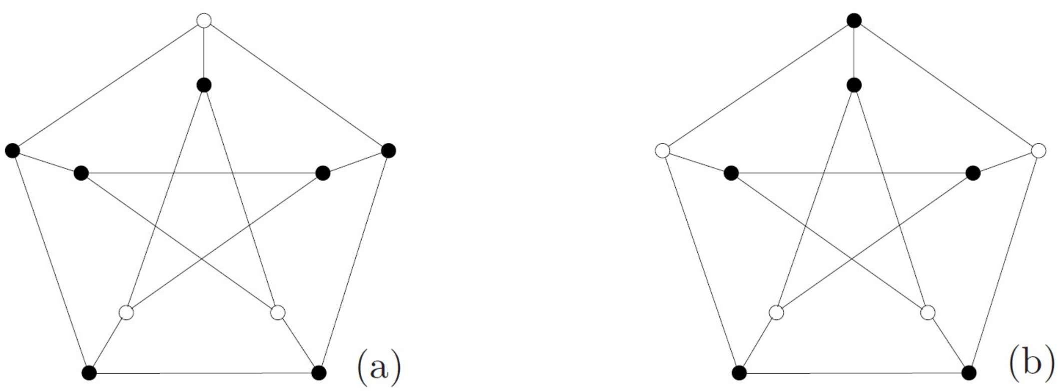

Fig. 16.7. (a) A minimal covering, (b) a minimum covering

Note. A generalization of Hall's Theorem (Theorem 16.4) is to be given in Exercise 16.2.9:

The König-Ore Formula.

The matching number of a bipartite graph $G = [X, Y]$ is given by

$$\alpha'(G) = |X| - \max\{|S| - |N(S)| \mid S \subseteq X\}.$$

Note. If M is a matching of G and K is a covering of G, then at least one end of each edge of M belongs to K. Now the ends are distinct in a matching, so $|M| \leq |K|$. It is to be shown in Exercise 16.2.2 that if equality holds then M is a maximum matching and K is a minimum covering, as follows.

Proposition 16.7. Let M be a matching and K a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

Note. The following result was stated in Section 8.6, "Linear and Integer Programming." In this section we give a proof (though a key step in the proof is based on Exercise 16.2.8).

Theorem 8.32. The König-Egerváry Theorem.
In any bipartite graph G, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

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