Section 16.3. Matchings in Arbitrary Graphs

Note. We now consider arbitrary graphs. We define hypomatchable graphs as those that are, in a sense, close to having a perfect matching. This leads to the idea of a "barrier." We show that every graph has a barrier in the Tutte-Berge Theorem (Theorem 16.11).

Note. For graph G, we denote by $o(G)$ the number of odd components of G (that is, the number of connected components on an odd number of vertices). If M is a matching of G, then each odd component of G must have at least one vertex of G that is not covered by matching M. With U as the set of vertices of G not covered by M, we have $|U| \geq o(G)$. This observation can be generalized to all induced subgraphs of G as follows.

Lemma 16.3.A. Let S be a proper subset of $V(G)$ and let M be a matching in G. Let U be the set of vertices of G not covered by M. Then

$$|U| \geq o(G - S) - |S|. \qquad (16.2)$$

Note 16.3.A. In Exercise 16.1.8(b) it was to be shown that the Sylvester graph of Figure 16.5 has no perfect matching. We can also show this by letting S be the set consisting of the single central vertex (so that $|S| = 1$). Then $G - S$ has three odd components so that $o(G - S) = 3$. So by Lemma 16.3.A the number of vertices not covered by M is $|U| \geq o(G - S) - |S| = 2$. A perfect matching covers all vertices, so no perfect matching exists.

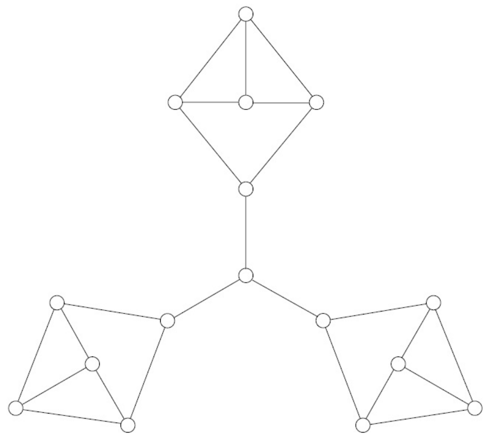

Fig. 16.5. The Sylvester graph: a 3-regular graph with no perfect matching

Similarly, the graph G of Figure 16.8(a) has no perfect matching, as shown by choosing S to be the three shaded vertices. $G-S$ has six components, five of which are odd. Hence $|S| = 3$, $o(G-S) = 5$ and by Lemma 16.3.A, $|U| = 5 - 3 = 2$. Therefore G has no perfect matching.

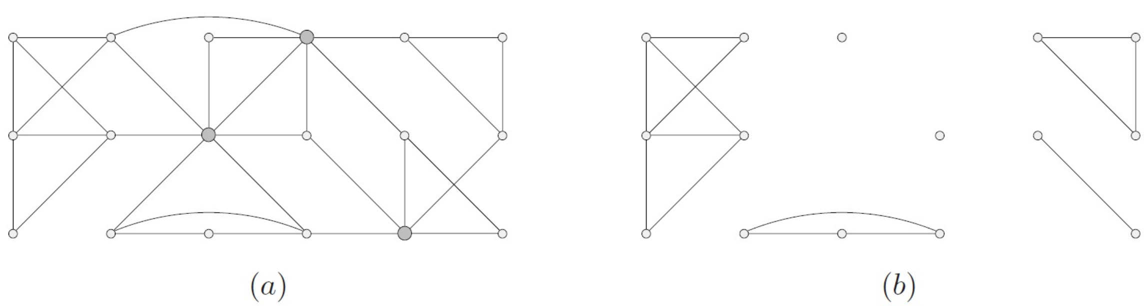

Fig. 16.8. A set S with $o(G-S) > |S|$

Note. If equality holds in Lemma 16.3.A, $|U| = o(G-B) - |B|$, for some $S = B \subset V(G)$ and some matching M of G, then B is a set of vertices for which the number of vertices not covered by M is minimal (notice that $|U| = v(G) - 2|M|$). This implies that M is a maximum matching, as is to be shown in Exercise 16.3.1.

So if such set of vertices B can be found for a given matching, then this insures that M is a maximum matching; in the terminology of Section 8.1. Computational Complexity, B "certifies" the optimality of matching M.

Definition. For graph G, if $B \subset V(G)$ is such that for some matching M of G we have $|U| = o(G - B) - |B|$ where U is the set of vertices of G not covered by M, then B is a *barrier* of G.

Note. As observed in Note 16.3.A, the graph G of Figure 16.8(a) has $o(G - S) = |S| = 2$ for S given by the shaded vertices. This, if we can find a matching on G where $|U| = 2$, then we know that B is a barrier of G. Such a matching is given in Figure 16.15(a).

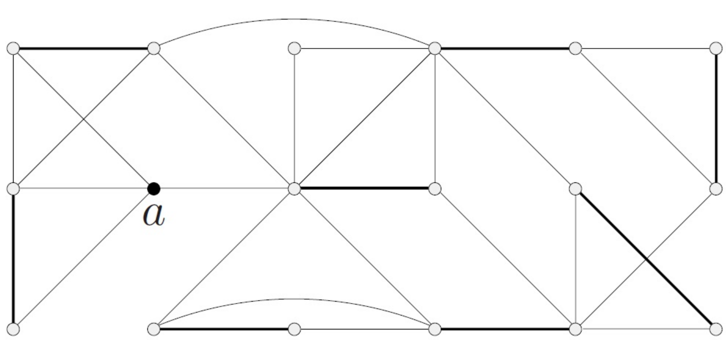

Fig. 16.15.(a)

Note 16.3.B. A graph with a perfect matching M (a "matchable graph") cannot have any odd components, and $U = \varnothing$. Hence, for such a graph $B = \varnothing$ is a barrier because

$$|U| = 0 = 0 - 0 = o(G) - |\varnothing| = o(G - \varnothing) - |\varnothing| = o(G - B) - |B|.$$

Also for $B = \{v\}$ (a single vertex) we have $U = \varnothing$ and $o(G - B) = 1$ (this claim is, a bit subtle and requires consideration of whether v is a cut vertex or not) so that $|U| = 0 = 1 - 1 = o(G - B) - |B|$ so that all singletons of a matchable graph are barriers. If a graph G has some vertex-deleted subgraph that is matchable with matching M, then graph G has $B = \varnothing$ as a barrier since $|U| = 1$ (U contains he vertex deleted from G to create the matchable graph) and $o(G - B) = 1$, so that $|U| = 1 = 1 - 0 = o(G - B) - |B|$. This idea of matchable subgraphs that result from a single-vertex deletion inspires the following definition.

Definition. A graph for which every vertex-deleted subgraph is matchable is *hypomatchable* or *factor-critical*.

Note 16.3.C. We'll explore the ideas of graph "factors" in the next section. The trivial graph K, is hypomatchable because deleting the single vertex leaves us a graph with no vertices (we have not disallowed such a thing from begin a graph) which is vacuously matchable. As argued in Note 16.3.B, all hypomatchable graphs have the empty set as a barrier. We formally state this next as a lemma. In fact, as is to be shown in Exercise 16.3.8, the empty set is the only barrier of a hypomatchable graph.

Lemma 16.8. The empty set is a barrier of every hypomatchable graph.

Note. The "fundamental theorem" concerning barriers is the fact that every graph has a barrier. This is the Tutte-Berge Theorem, to be stated below. Exercise 16.3.7(a) requires a proof of the Tutte-Berge Theorem based on induction on the number of vertices. The base case is given for the trivial graph K_1 by Lemma 16.8 since $\varnothing$ is a barrier in this case. The next definition was first encountered in Exercise 16.1.15.

Definition. A vertex v of graph G is *essential* if every maximum matching of G covers v, and inessential otherwise.

Note. In a path of length 2, the center vertex is essential and the end-vertices are inessential. Notice that f v is essential vertex of G then the matching number satisfies $\alpha'(G - v) = \alpha'(G) - 1$, and if v is inessential if $\alpha'(G - v) = \alpha'(G)$. The proof of the following is to be given in Exercise 16.3.5.

Lemma 16.9. Let v be an essential vertex of a graph G and let B be a barrier of $G - v$. Then $B \cup \{v\}$ is a barrier of G.

Note. To show that every graph has a barrier, by Lemma 16.9 it suffices to consider graphs with no essential vertices. This is done in the next lemma for connected graphs. It is to be extended to the general case in Exercise 16.3.6.

Lemma 16.10. Let G be a connected graph, no vertex of which is essential. Then G is hypomatchable.

Note. We now claim that every graph has a barrier. This is to be proved in Exercise 16.3.7(a) by induction on the number of vertices. The corollary the follows from it is to be proved in Exercise 16.3.7(b).

Theorem 16.11. The Tutte-Berge Theorem.
Every graph has a barrier.

Corollary 16.12. The Tutte-Berge Formula.
For any graph G

$$\alpha'(G) = \frac{1}{2}\min\{v(G) - (o(G-S) - |S|) \mid S \subset V\}.$$

Note. The Tutte-Berge Theorem can be refined to give that every graph G has a barrier such that each off component of $G - B$ is hypomatchable and each even component of $G - B$ has a perfect matching. Such a barrier is called a *Gallai barrier*. This idea is explored more in Section 16.5, "Matching Algorithms."

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