Chapter 17. Edge Colourings

Section 17.1. Edge Chromatic Number

Note. Just as we considered colouring vertices in Chapter 14, we now consider assigning colours to edges of a graph. We will see that many of the ideas from the vertex setting (such as chromatic number) carry over to analogous ideas in the edge setting.

Definition. A *k-edge colouring* of graph $G = (V, E)$ is a mapping $c : E \to S$, where S is a set of k *colours* (usually denoted $S = \{1, 2, \ldots, k\}$). An edge colouring is *proper* if adjacent edges receive distinct colours. A graph is *k-edge colourable* if it has a k-edge-colouring. The *edge chromatic number*, denoted $\chi'(G)$, of a graph G is the minimum k for which G is k-edge colourable, and G is *k-edge-chromatic* if $\chi'(G) = k$.

Note 17.1.A. Just as the colour classes of a vertex colouring of graph G yield stable sets (and also a partition of the vertex set; see Note 14.1.A), a similar idea holds for edge colourings. A k-edge-colouring results in a partition $\{E_1, E_2, \ldots, E_k\}$ of edge set E, where E_i denotes the (possible empty) set of edges assigned colour i. Though we have defined an edge colouring as a mapping, we commonly discuss edge colourings in terms of this type of partitioning. In a proper k-edge-colouring the edge set partition has each E_i as a matching of G. A graph with loops does not have a proper edge colouring. **In this chapter, all graphs are loopless and a proper edge colouring is referred to as simply an "edge colouring."**

Note. The graph G of Figure 17.1 with edge set $\{a, b, c, d, e, f\}$ is f-edge-colourable where the partition of the edge set based on the 4-colours is $\{\{a, g\}, \{b, e\}, \{c, f\}, \{d\}\}$. It is to be shown in Exercise 17.1.3 that this graph is not 3-edge colourable. Therefore $\chi'(G) = 4$. In any graph G, there are Δ edges incident to some single vertex of G, so that a (proper) edge colouring of G requires at least Δ colours. That is, $\chi' \geq \Delta$.

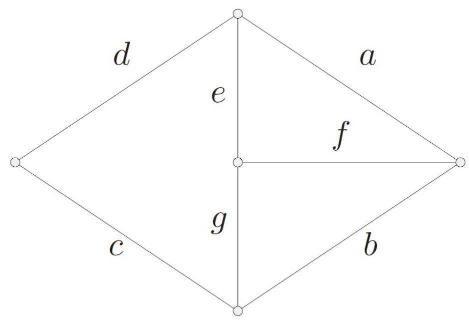

Fig. 17.1. A 4-edge-chromatic graph

Example 17.1. The Timetabling Problem.

Consider the problem in a school of assigning m teachers $x_1, x_2, \ldots, x_m$ to n classes $y_1, y_2, \ldots, y_n$. The goal is to minimize the number of periods required to schedule the classes. Let p_{ij} denote the number of periods that teacher x_i is assigned to teach class y_j (at ETSU, this is the teaching "work load" of a faculty member). We translate this into graph theory by introducing bipartite graph $H[X, Y]$ where $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and vertices x_i and y_j are joined by p_{ij} edges. In Exercise 17.1.10(a) it is to be shown that finding a minimum number of periods is equivalent to finding a (proper) k-edge-colouring of $H[X, Y]$ for smallest k.

Note. As discussed above, we easily have the lower bound of Δ on the edge chromatic number χ'. Below, we show that this lower bound is attained by every bipartite graph. We'll see in the next section that for simple graphs $\chi' \leq \Delta + 1$ (in Vizing's Theorem, Theorem 17.4). Our proofs of these results are contructive and require us to introduce the following definitions.

Definition. Let H be a spanning subgraph of graph G and let $\mathcal{C} = \{M_1, M_2, \ldots, M_k\}$ be a k-edge-colouring of H. Colour i is *represented* at a vertex v if it is assigned to some edge of H incident with v; otherwise it is *available at vertex* v. A colour is *available for an edge* of $E(G) \setminus E(H)$ if it is available at both ends of the edge.

Note/Definition 17.1.B. In the notation of the previous definition, edge $e \in E(G) \setminus E(H)$ may be assigned any colour available to it to extend colouring $\mathcal{C}$ to a k-edge-clouring of $H + e$ (this the term "available"). For i and j two distinct colours, define $H_{ij} = H[M_i \cup M_j]$. By Note 17.1.A, M_i and M_j are (edge- disjoint matchings and $M_i \triangle M_j = M_i \cup M_j$ so that $H[M_i \cup M_j] = G[M_i \triangle M_j]$. As seen in the proof of Berge's Theorem (Theorem 16.3) the components of H_{ij} are even length cycles and paths. The path-components of H_{ij} are *ij-paths.* We use ij-paths in the proof of the following, which gives the edge chromatic number of a bipartite graph.

Theorem 17.2. If G is bipartite, then $\chi' = \Delta$.

Note. Bondy and Murty claim that the proof technique of Theorem 17.2 can be used to "easily extract" a polynomial-time algorithm for finding a Δ-edge-colouring of a bipartite graph (see page 459).

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