Section 17.2. Vizing's Theorem

Note. In this section, we state what is probably the best-known colouring theorem (other than the Four-Colour Theorem). We prove that for G a simple graph, either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$. This result is Vizing's Theorem (Theorem 17.4). This allows us to partition simple graphs into two classes. We also present a related result for loopless nonsimple graphs (Theorem 17.5).

Note. Vizing's Theorem is due to Vadim Vizing and appears in "On an Estimate of the Chromatic Class of a p-Graph," *Diskret. Analiz.* (the full name of the journal is *Akademiya Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Diskretnyi Analiz. Sbornik Trudov,*) **3**, 25–30 (1964). The paper was published in Russian. The result was independently shown by R. P. Gupta while doing his dissertation work; his contribution seems limited to an abstract published in the *Notices of the American Mathematical Association* in 1966 (Abstract 66T-429). As in the proof that bipartite graphs have edge chromatic number Δ (Theorem 17.2), we'll give a proof of Vizing's Theorem based on induction on the number of edges by extending edge colourings from $G \setminus e$ to G. The bulk of the argument is contained in the following lemma. In the proof of the lemma, we go through seven structures. We'll call attention to each of these structures and illustrate them with an example following the proof.

Lemma 17.3. Let G be a simple graph, v a vertex of G, e an edge of G incident to v, and k an integer with $k \geq \Delta$. Suppose that $G \setminus e$ has a k-edge-colouring c with respect to which every neighbor of v has at least one available colour. Then G is k-edge-colourable.

Note. We now illustrate the bold words in the proof of Lemma 17.3.

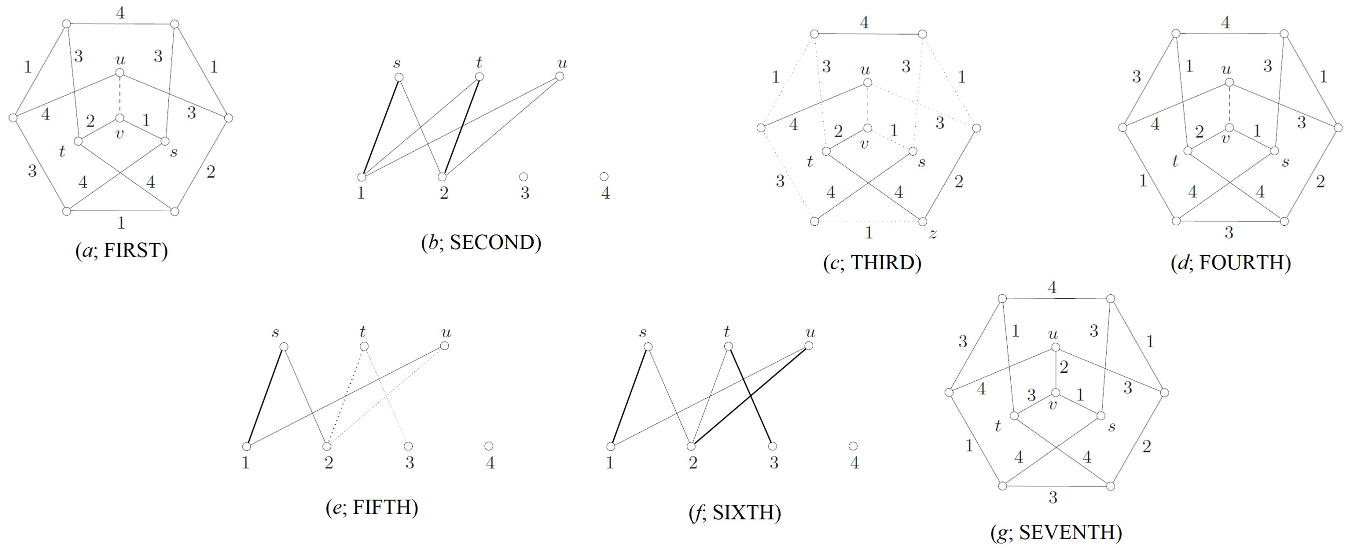

Fig. 17.2. Finding a 4-edge-colouring of the Petersen graph by means of Lemma 17.3

In Figure 17.2(a), a drawing of the Petersen graph is given with a 4-edge-colouring (notice $\Delta + 1 = 4$), except for edge $e = vu$. This is the initial k-edge-colouring (**FIRST**). With $X = \{s, t, u\}$ (the neighbors of v in G) and $Y = \{1, 2, 3, 4\}$, we get **SECOND** the bipartite graph $H = H[X, Y]$ given in Figure 17.2(b). Matching M is given by the heavy lines. We take $i = 1$, $j = 3$ (a colour available at v), $u = x$, and $t = y$ (colour $i = 1$ is then adjacent to both x and y in H, and colour $j = 3$ is represented at both s and t in $G \setminus e$). Notice from H that we have (because of the M-alternating path definition) $Z = \{s, t, u, 1, 2\}$, $R = \{s, t, u\}$ and $B = \{1, 2\}$; also, $N_H(R) = B$. The ij-path P (**THIRD**) from v to u and the ij-path$P$ from t to z are given in Figure 17.2(c) using dotted lines (notice these paths have edges alternating between colours $i = 1$ and $j = 3$). **Fourth**, the 4-edge-colouring c' is given in Figure 17.2(d) where the colours 1 and 3 of path P (from $y = t$ to z) are interchanged. The new bipartite graph $H' = H'[X, Y]$ is given **FIFTH** in Figure 17.2(e) (in this case, only one edge incident to $t = y$ has changed from H). The

M-augmented ut-path (here $ut = uy$) Q (with vertices $u, 2, 1$) and its extension Q' to include $j = 3$ (given by the dotted edges; notice that edge $(s, 1)$ is still bold, as it was in M). **SIXTH** in Figure 17.2(f) we have created matching M' of H' by replacing edge $(t, 2)$ in M with edges $(u, 2)$ and $(t, 3)$ (notice these three edges make up the M-augmented path Q' and we are choosing ever-other edge of Q', picking up one new edge). Finally, **SEVENTH** we use the augmented matching M' to assign colours to edges incident to v. Edge vs is assigned colour 1, edge vt is assigned colour 3, and edge uv is finally coloured with colour 2. This gives us a 4-edge-colouring of G, the Petersen graph.

Note. We see from the proof of Lemma 17.3 and Figure 17.2 that this is just an elaborate reshuffling of the colouring of $G \setminus e$ to insure that some colour is available to be assigned to edge $e = vu$. The key step is the introduction of the "augmented matching" M', which is where we pick up the needed colour to assign to the "extra" edge e. This is similar to the idea used in the proof of Theorem 17.2 ($\chi'(G) = \Delta$ for G bipartite), but much more complicated. We now have the equipment to give a quick, inductive proof of Vizing's Theorem.

Theorem 17.4. Vizing's Theorem.
For any simple graph G, $\chi' \leq \Delta + 1$.

Note. We might seek a simpler proof of Vizing's Theorem (that is, one that is not based on Lemma 17.3). In my online notes for the senior/graduate level Introduction to Graph Theory (MATH 4347/5347), Vizing's Theorem is only stated for

k-regular graphs (and a proof is not given even in this special case); see my notes on Section 2.2. Edge Colorings and notice Theorem 2.2.2. In Bondy and Murty's *Graph Theory with Applications*, Macmillan Press (1976), also a book meant for a senior/graduate level class, a proof by contradiction is given of Vizing's Theorem (see Chapter 6 "Edge Colourings," Section 6.2 "Vizing's Theorem," Theorem 6.2; the lemmas of Section 6.1, "Edge Chromatic Number," are also required for the proof). This proof is based on J. C. Fournier's "Colorations des Arêtes d'un Graphe," *Cahiers du CERO*, **15**, 311-314 (1973). It avoids the introduction of the bipartite graph $H = H[X, Y]$ in our proof. It seems that in our text book, the authors have presented the constructive proof so that they can address algorithms for finding $(\Delta+1)$-edge-colourings. They claim (see page 463) that: "The above proof [of Lemma 17.3] yields a polynomial-time algorithm for finding a k-edge-colouring of a simple graph G, given a k-edge-colouring of $G \setminus e$ satisfying the hypothesis of Lemma 17.3." In Exercise 17.2.6, a polynomial-time algorithm for finding a proper $(\Delta + 1)$-edge-colouring of a simple graph G is to be given. The solution to this exercise involves building up G one edge at a time and inductively applying the algorithm for colouring G based on a colouring of $G \setminus e$. We consider the topic of graph algorithms in Mathematical Modeling Using Graph Theory (MATH 5870); see my online notes for this material based on Chapters 6, 7, 8, 20, and 21 of our text book.

Note. Vizing's Theorem applies to simple graphs. Of course, we can't address edge-colourings for graphs with loops. However, we can address chromatic numbers for loopless graphs with multiple edges. We need to introduce another parameter for such graphs.

Definition. Let G be a loopless graph. For vertices u and v, et $\mu(u, v)$ denote the number of parallel edges joining u and v. The *multiplicity* of G, denoted $\mu(G)$, is the maximum value of μ taken over all pairs of vertices:

$$\mu(G) = \max\{\mu(u, v) \mid u, v \in E, u \neq v\}.$$

If $\mu(G) > 1$ then G is a *multigraph.*

Note. In a mulitgraph G, it is common to refer to the "mutliset" of edges. In a multiset, elements come with a multiplicity (normally represented by repetition of the element an appropriate number of times). The next result concerns the chromatic number of multigraphs. It is also due to Vadim Vizing and appears in "The Chromatic Class of a Multigraph," *Kibernetica* (*Kiev*), **3**, 29–29 (1965) (in Russian).

Theorem 17.5. For any loopless graph G with multiplicity μ, we have $\chi' \leq \Delta + \mu$.

Note. A proof of Theorem 17.5 is to be given in Exercise 17.2.8 by adapting the proof of Vizing's Theorem (Theorem 17.4) to multigraphs. The result is best possible (or "sharp") as shown by the graph G of Figure 17.3 (the μ-fold K_3; see below). Here, $\delta = 2\mu$ and $\chi' = m = 3\mu$, so that $\chi' = 3\mu = 2\mu + \mu = \Delta + \mu$.

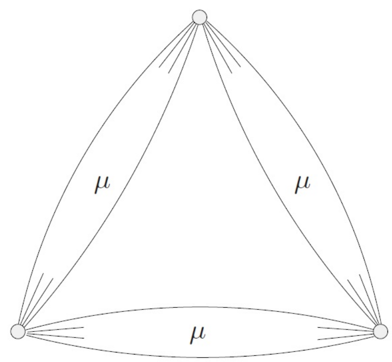

Fig. 17.3. A graph G with $\chi' = \Delta + \mu$

Note. Again considering simple graphs, we see that Vizing's Theorem partitions such graphs into two classes.

Definition. A simple graph for which $\chi' = \Delta$ is *Class 1.* A simple graph for which $\chi' = \Delta + 1$ is *Class 2.*

Note. It is known that the problem of deciding if a graph is Class 1 or Class 2 is $\mathcal{NP}$-hard. This was shown in:

1. Ian Holyer, "The NP-Completeness of Edge-Coloring," *SIAM Journal on Computing*, **10**(4), 718–720 (1981).

2. D. Leven and Z. Galil, "NP Completeness of Finding the Chromatic Index of Regular Graphs," *Journal of Algorithms*, **4**, 35-44 (1983). This can be viewed online from Stéphane Bessy's LIRMM page.

Thus results for graphs with certain properties which determine their class would be useful. For example, in Exercise 17.2.9 it is to be shown that if the vertices of degree Δ of simple graph G induce a forest then G is Class 1. In Exercise 17.2.1, it is to be shown that a simple graph satisfying $m > \lfloor n/2 \rfloor \Delta$ is Class 2.

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