Section 17.3. Snarks

Note. In this section, we recall some previous definitions and use them to define snarks. We give some examples and briefly describe their relationship to the Double Cover Conjecture and the Four-Colour Theorem.

Note/Definition. Recall that a *cubic graph* is a 3-regular graph. As shown in Exercise 17.1.12, such a graph has edge chromatic number three or four (whether the graph is simple or not; if it is simple then this observation follows from Vizing's Theorem). Also recall from Section 9.3. Edge Connectivity that a nontrivial graph G is *k-edge-connected* if for any two distinct u and v vertices of G, the maximum number of pairwise edge disjoint uv-paths is at least k. A k-edge connected graph is *essentially* $(k+1)$*-edge connected* if all k-edge cuts (that is, edge curs $\partial(X)$ where $\varnothing \subsetneq X \subsetneq V$ and $|\partial(X)| = k$) are trivial (that is, ar associated with a set X containing one vertex and so are of the form $\partial(\{x\})$). In Note 9.3.B, it is shown that cubic graph $K_{3,3}$ is essentially 4-edge-connected but cubic graph $K_3 \square K_2$ is not essentially 4-edge-connected. We need these ideas for our definition of a snark.

Definition. A 4-edge-chromatic essentially 4-edge-connected cubic graph is a *snark*.

Note. Since a snark satisfies $\Delta = 3$ and $\chi' = 4 = \Delta + 1$, then snarks are Class 2 graphs.

Note. Essentially 4-edge-connected cubic graphs play a role in the Cycle Double Cover Conjecture (Conjecture 3.9). One can show that to prove the conjecture it is sufficient to prove it for essentially 4-edge-connected cubic graphs (by sewing together an argument based on Theorem 5.5, Exercise 9.3.9, and Exercise 9.4.2). In addition, if such a graph is 3-edge-colourable then by Exercise 17.3.4(a) it admits a covering by two even subgraphs and hence (by Exercise 3.5.4(a) it has a cycle double cover. This it suffices to establish the Cycle Double Cover Conjecture for essentially 4-edge-connected cubic graphs that are NOT 3-edge-colourable (that is, if the Cycle Double Conjecture can be proved for snarks, then it follows in general).

Note. It seems that the definition of a snark is not universal. For example, in Introduction to Graph Theory (MATH 4347/5347) a snark is defined as a cubic graph with edge chromatic number four (with no regard for the connectivity); see my online notes for this class on Section 2.2. Edge Colorings. It is to be shown in Exercise 17.3.1 that the Petersen graph is the smallest snark. The Blasnuša snark on 18 vertices is given in Figure 17.4(b).

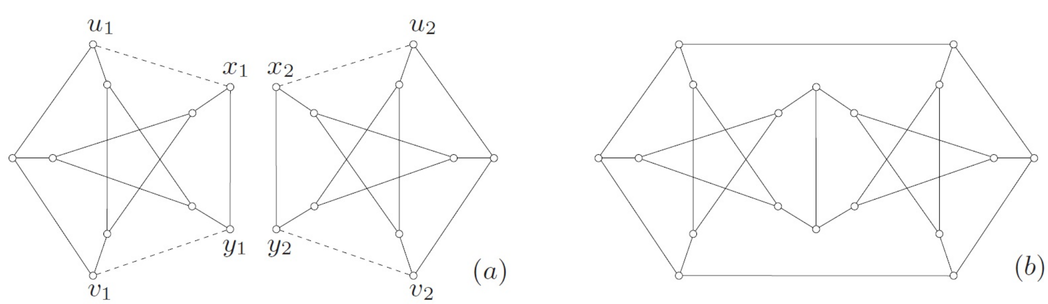

Fig. 17.4. Construction of the Blanuša snark

The first to introduce an infinite class of snarks was Rufus Isaacs in "Infinite Families of Nontrivial Trivalent Graphs which are not Tait Colorable," *American Mathematical Monthly*, **82**, 221–239 (1975); this is posted online on the the JSTOR website (accessed 7/8/2022). Isaacs' graphs are called "flower snarks" (see Exercise 17.3.3); see Figure 17.5 for his flower snark on 20 vertices.

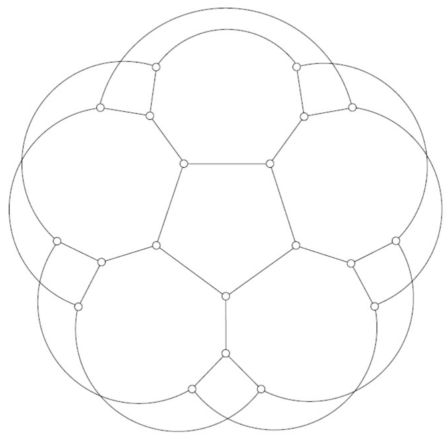

Fig. 17.5. A flower snark on twenty vertices

Note. William Tutte conjectured in "On the Algebraic Theory of Graph Colorings," *Journal of Combinatorial Theory*, **1**, 15-50 (1966) (a copy can be viewed online on the ScienceDirect website; accessed 7/8/2022) that every snark has a Petersen graph minor. If this can be proved, then the result can be used (with Tait's Theorem, Theorem 11.5) to prove the Four-Colour Theorem. In fact Tutte's Conjecture was confirmed by N. Robertson, D. Sanders, P.D. Seymour, and R. Thomas, as explained "Tutte's Edge-Colouring Conjecture," *Journal of Combinatorial Theory-B*, **70**, 166-183 (1997); a copy is online on ScienceDirect website (accessed 7/8/2022). Unfortunately, their approach used the same sorts of techniques used in the proof of the Four-Colour Theorem of Appel, Haken, and Koch in 1977 (as described in Section 15.2. The Four-Colour Theorem. So this gives an alternative proof of the Four-Colour Theorem, but it is no more clear or efficient than the original 1977 proof.

Note. Bondy and Murty conclude this brief section with the comment (see page 468: "…the general structure of snarks remain a mystery."

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