Section 17.4. Coverings by Perfect Matching

Note. In this very brief section, we consider edge coverings of regular graphs by perfect matching. We mention 1-coverings (that is, edge decompositions) and 2-coverings (that is, double covers).

Note. A k-edge-colouring of a k-regular graph (is such an edge-colouring exists) is an edge decomposition of the graph into k perfect matching (notice also that such a graph is Class 1). Conversely, a decomposition of a graph into perfect matchings only exists for regular Class 1 graphs. A decomposition is a 1-cover of the edge set. A related problem then is to consider regular graphs that admit uniform coverings (we concentrate on 2-coverings/double coverings).

Note. In Exercise 17.4.6 a condition is given for a k-regular graph which (it is to be shown) insures a uniform covering (that is, a covering where each edge is covered the same number of times) by perfect matchings. In 1971, D. R. Fulkerson "[m]otivated by certain questions concerning the polyhedra defined by the incidence vectors of perfect matchings" (see page 470) conjectured the following (notice the similarity to the Cycle Double Cover Conjecture, Conjecture 3.9).

Conjecture 17.6. Fulkerson's Conjecture.

Every 2-connected cubic graph admits a double cover by six perfect matchings.

Note. Cubic graphs of Class 1 are 3-edge-colourable (by the definition of Class 1) so that the edge set can be partitioned using the colours of the edges as $\{M_1, M_2, M_3\}$. Each M_i is a perfect matching and $\{M_1, M_2, M_3\}$ is a 1-cover of the graph. So we can simply take two copies of each M_i to get a double cover with six (not distinct) perfect matchings. So the conjecture needs only to be established (or violated) for Class 2 graphs. In Exercise 17.4.1, it is to be shown that the Class 2 Petersen graph satisfies Fulkerson's Conjecture.

Note. If Fulkerson's Conjecture is true and we shift out attention to covers where each edge is covered *at least once* instead of double covers, then we see that we can drop one of the six perfect matchings to get a cover. That is, Fulkerson's Conjecture implies the following weaker conjecture of Claude Berge.

Conjecture 17.8. Every 4-connected cubic graph admits a covering by five perfect matchings.

Note. In Exercise 17.4.4(a) it is to be shown that the Petersen graph admits no covering by fewer than five perfect matchings. So Conjecture 17.7 cannot hold for four or less perfect matchings.

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