Section 17.5. List Edge Colourings

Note. In this section, we extend the idea of a list colouring from vertex colourings to edge colourings. We consider a relation between the edge colouring number and the list edge colouring number of a loopless graph in the List Edge Colouring Conjecture. We give a proof in the conjecture in the special case of a simple bipartite graph (Theorem 17.10).

Definition. Let G be a graph and let L be a function which assigns to each edge e of G a set $L(e)$ of positive integers, called the *list* (of colours) of e. An edge colouring $c : E \to \mathbb{N}$ such that $c(e) \in L(e)$ for all $e \in E$ is a *list edge colouring* of G with respect to L, or an *L-edge-colouring*, and G is said to be *L-edge-colourable*. Graph G is *k-list-edge-colourable* if it has a list edge colouring whenever all the lists have length k. The smallest value of k for which G is k-list edge colourable is the *list edge chromatic number* denoted $\chi'_L(G)$.

Note 17.5.A. For any loopless graph we certainly have $\chi'_L(G) \geq \chi'(G)$. In fact, for vertex colourings we had the relationship between the list chromatic number and the chromatic number: $\chi_L(G) \geq \chi(G)$. We saw in Figure 14.9 that this inequality can be strict. However, it is conjectured that equality holds for the list edge chromatic number and the list chromatic number, as follows.

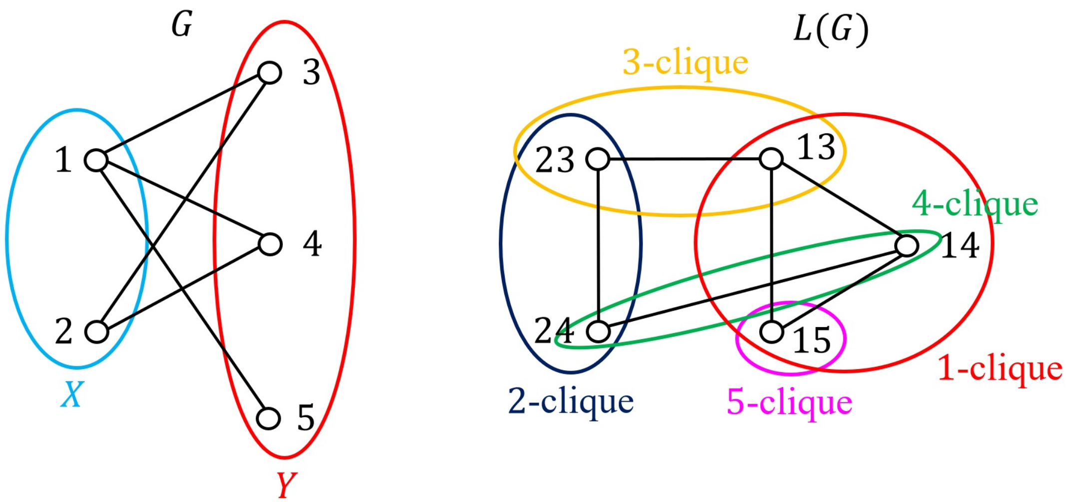
Conjecture 17.8. The List Edge Colouring Conjecture.
For every loopless graph G, $\chi'_L(G) = \chi'(G)$.

Note. This conjecture seems to have been floating around the chromatic graph theory community in the early 1980's. It first appeared in print in B. Bollabás and A. J. Harris' "List-Colourings of Graphs," *Graphs and Combinatorics*, **1**, 115-127 (1985).

Note. The List Edge Colouring Conjecture has been proved in the special case of bipartite graphs. This was proved by Fred Galvin in "The List Chromatic Index of a Bipartite Multigraph," *Journal of Combinatorial Theory-B*, **63**, 153–158 (1995); a copy is online on Alexandr Kostochka's University of Illinois at Urbana-Champaign webpage (accessed 7/8/2022).

Note 17.5.B. Recall from Section 1.3. Graphs Arising from Other Structures that the line graph $L(G)$ of graph G is the graph with vertex set $\{ij \mid ij \in E(G)\}$ with vertices ij and $k\ell$ adjacent in $L(G)$ if edges ij and $k\ell$ share a vertex in G. Hence colouring the edges of a graph G amounts to colouring the vertices of its line graph $L(G)$. Galvin uses this idea in his proof.

Note/Definition. Let $G = G[X, Y]$ be a simple bipartite graph. In the line graph $L(G)$ there is a clique K_v for each vertex v of G (the vertices of K_v corresponding to the edges of G incident to v). Each edge xy of G corresponds to a vertex of $L(G)$ which lies in exactly two of these cliques, namely K_x and K_y. The clique K_v is an X-*clique* if $v \in X$ and a Y-*clique* if $v \in Y$. See the figure below.



Note 17.5.C. We now describe how to visualize a line graph. For graph G, the vertices of $L(G)$ are (unordered) pairs of vertices of G. So we can visualize the vertex set of $L(G)$ as the set $X \times Y$ (though vertex $xy \in V(G)$ is then associated with both (x, y) and (y, x); this will give a visualization that is symmetric). We create an $m \times n$ grid in the Cartesian plane, where $m = |X|$ and $n = |Y|$, with the rows of the grid indexed by X and the columns indexed by Y. Adjacent vertices in $L(G)$ share either a first coordinate or a second coordinate. Thus vertices that lie in the same row or in the same column of the grid are adjacent in $L(G)$. Therefore the rows of the grid represent the X-cliques and the columns represent the Y-cliques. See Figure 17.7 for an example of such a grid.

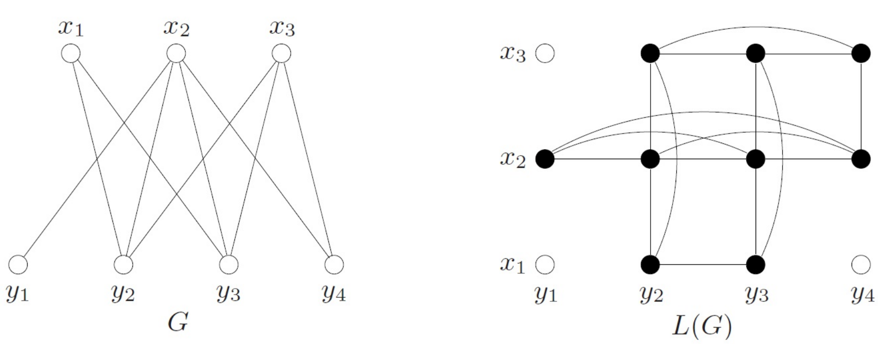


Fig. 17.7. Representing the line graph $L(G)$ of a bipartite graph G on a grid

Note. A topic covered in a design theory class is Latin squares. See my online notes for Design Theory (not an official ETSU class); notice in particular Chapter 6, "Mutually Orthogonal Latin Squares." A Latin square of order n is an $n \times n$ array of n symbols such that each symbol occurs exactly once in each row and exactly once in each column. The Cayley table for an order n group gives an example of a Latin square of order n. In Exercise 17.5.1 it is to be shown that there is a one-to-one correspondence between n-edge-colourings of $K_{n,n}$ (in colours $1, 2, \ldots, n$) and Latin squares of order n (on the symbols $1, 2, \ldots, n$).

Note. We need a preliminary theorem before proving Galvin's Theorem. We orient the line graph of a simple bipartite graph in such a way that the resulting digraph has a kernel. Recall from Section 12.1. Stable Sets that a kernel of a digraph D is a stable set S of D (that is, a stable set of the underlying graph of D) such that each vertex of $D - S$ dominates some vertex of S (that is, for each vertex u of $D - S$ there is a vertex of v of S such that (u, v) is an arc of D).

Theorem 17.9. Let $G[X, Y]$ be a simple bipartite graph, and let D be an orientation of its line graph $L(G)$ in which each X-clique and each Y-clique induces a transitive tournament. Then D has a kernel.

Theorem 17.10. Every simple bipartite graph G is Δ-list-edge-colourable.

Note. Galvin's general theorem does not have the condition that G is simple. Galvin's Theorem states that: "Every bipartite graph G is Δ-list edge-colourable. This is to be shown in Exercise 17.5.3.

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