Section 17.6. Related Reading

Note. In this section, we state the Total Colouring Conjecture, and mention some of the few results concerning it.

Definition. A *total colouring* of a graph G is a colouring $c : V \cup E \to S$, where S is a set of *colours*. The colouring c is *proper* if

1. its restriction to V is a proper vertex colouring of G,

2. its restriction to E is a proper edge colouring of G, and

3. no edge receives the same colour as either of its ends.

The *total chromatic number* of G, denoted by $\chi''(G)$, is the minimum number of colours in a proper total colouring of G.

Note 17.6.A. Vadim Vizing in "On an Estimate of the Chromatic Class of a p-Graph," *Diskret. Analiz.*, **3**, 25–30 (1964) [in Russian], the same paper in which he proved Vizing's Theorem (Theorem 17.4), conjectured that the total chromatic number of a graphs is at most $\Delta + 2$. Independently, Mehdi Behzad made the same conjecture in his Ph.D. dissertation *Graphs and Their Chromatic Numbers* (1965) from Michigan State University (the abstract, table of contents, and first six pages can be viewed on ProQuest; accessed 7/8/2019). At a vertex v of degree Δ, we need Δ colours for the proper edge colouring and we need one additional colour for vertex v, so we immediately have that $\chi'' \geq \Delta + 1$. Hence, if the conjecture holds then we have that $\chi''(G)$ is either $\Delta + 1$ of $\Delta + 2$. Formally, we have the following.

Conjecture 17.A. The Total Colouring Conjecture.
For every loopless graph G, $\chi''(G) \le \Delta + 2$.

Note. Some early progress is due to M. Behzad, G. Chartrand, and J. K. Cooper and appeared in "The Colour Numbers of Complete Graphs," *Journal of the London Mathematical Society*, **42**, 226–228 (1967). A copy is online on the Radboud Universiteit website. They considered complete graphs and complete bipartite graphs. They proved that $\chi''(K_{n,n}) = \Delta + 2$, and for $m \ne n$ that $\chi''(K_{m,n}) = \Delta + 1$ (which we state and prove below). For complete graphs, they showed that

$$\chi''(K_n) = \begin{cases} \Delta + 1 & \text{if } n \text{ is odd} \\ \Delta + 2 & \text{if } n \text{ is even.} \end{cases}$$

They also proved an interesting result involving a relationship between the (vertex) chromatic number, edge chromatic number, and total chromatic number. They proved that if simple graph G has at least two vertices and $\chi(G) + \chi'(G) = \chi''(G)$, then G is bipartite. Surprisingly, this reference is not mentioned in Bondy and Murty.

Theorem 17.6.A. For $m \ne n$, $\chi''(K_{m,n}) = \Delta + 1$. In addition, $\chi''(K_{n,n}) = \Delta + 2$.

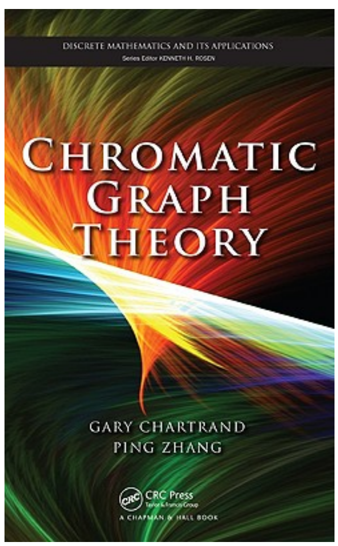
Note. There seems to be only limited progress made on this conjecture. In 1998 Michael Molloy and Bruce Reed used probabilistic methods to show that for $\Delta(G)$ sufficiently large, $\chi''(G) \le \Delta + 10^{26}$. Granted 10^{26} is a long way from 2, but this seems to be the most general result concerning the conjecture. Their result appeared in "A Bound on the Total Chromatic Number," *Combinatorica*, **18**, 241–

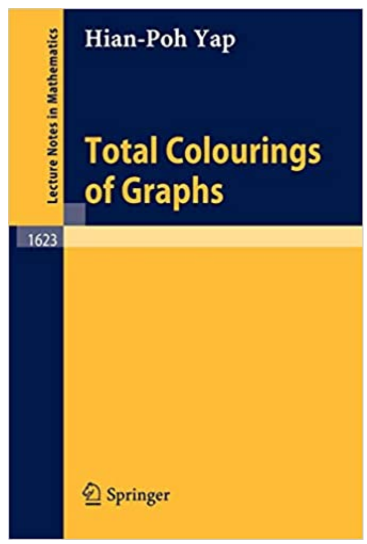
280 (1998); a copy can viewed online on the Springer website. Hugh Hind, Molloy, and Reed in 1998 also gave a bound of $\chi''(G) \le \Delta(G) + 8\ln^8(\Delta(G))$. Their result appeared in "Total coloring with $\Delta + \mathrm{poly}(\log\Delta)$ colors." *SIAM Journal on Computing*, **28**(3), 816-821 (1998). Now $8\ln^8(\Delta(G))$ is also potentially a long way from 2; in fact, as Δ increases, this bound increases without bound (granted, slowly because of the logarithm function).

Note. On of my graduate colleagues at Auburn University in the Department of Algebra, Combinatorics, and Analysis ("ACA"), Erin Spicer, did a master's thesis titled "On the Total Coloring Conjecture" (1991), 36 pp. To my knowledge, this included a survey of results on the conjecture up through 1991. By the way, she went on to complete a Ph.D. at Auburn with a dissertation titled "Graph Designs: With and Without Subsystems" (1995) 143 pp.

Note. If you Google "proof of the total coloring conjecture," you may find some manuscripts alleging a proof. In particular, you might find something posted on arXiv.org. This is a common occurrence and I have seen a few manuscripts posted on arXiv that have not made it through the peer review process (sometimes the manuscripts have been out there for several years). No disrespect to the manuscripts out there (it *is* nice to see what people are working on), but until a manuscript has survived peer review, its contents are not part of the mathematical body of knowledge.

Note. A major source on vertex/edge/total graph colourings is Gary Chartrand and Ping Zhang's *Chromatic Graph Theory*, CRC Press (2009). Notice that this was published a year after Bondy and Murty's text book, explaining why they do not reference it. The first five chapters cover general introductory graph theory. Chapter 6 introduces vertex colourings, Chapter 7 considers bounds on the chromatic number, Chapter 8 covers the Four Color Problem and chromatic polynomials, Chapter 9 includes list colourings, and Chapter 10 considers edge colourings and total colourings. The remaining four chapters cover related topics (rainbow colourings, complete colourings, and the relationship between colourings and domination).





Finally, we mention the more specific (and particularly relevant to topics mentioned above) source, Hian-Poh Yap's *Total Colourings of Graphs*, Springer-Verlag Lecture Notes in Mathematics #1623 (1996). This brief book (131 pages) focuses on the topic given in the title. It includes exercises could be used as a special topics text book.

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