Chapter 18. Hamilton Cycles

Section 18.1. Hamiltonian and Nonhamiltonian Graphs

Note. In this section, we give a condition which, when satiafied, shows that a graph is not hamiltonian. Inspired by the Petersen graph, we introduce the class of hypohamiltonain graphs.

Note. Recall from Section 2.2. Spanning and Induced Subgraphs that a spanning path of a graph is called a Hamilton path, and a spanning cycle is called a Hamilton cycle.

Definition. A graph is *traceable* if it contains a Haliton path. A graph is *non-hamiltonian* if it contains a Hamilton cycle.

Note. Of course a hamiltonian graph is traceable (just drop an edge from the Hamilton cycle). However, a traceable graph need not be hamiltonian. In Figure 18.1(a) the hamiltonian dedecahedron is give; in Fugire 18.1(b) the nonhamiltonian but traceable Herschel graph is bipartite but, but one partite set has an odd number of vertices so that it cannot be hamiltonian (a Hamilton cycle would have to have an odd number of edges).

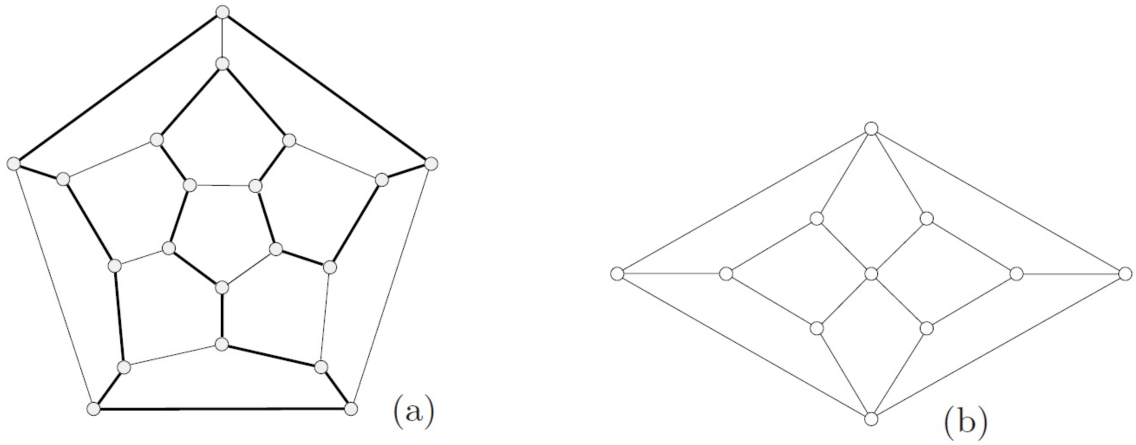

Fig. 18.1. Hamiltonian and nonhamiltonian graphs: (a) the dodecahedron, (b) the Herschel graph

Note. In Exercise 8.3.5(b) it is to be shown that the problem of deciding whether or not a graph is hamiltonian is $\mathcal{NP}$-complete. Therefore a condition that is easily checked (or at least reasonably checked) which guarantees that a graph is hamiltonian (or guarantees that it is nonhamiltonian) is useful. We now state a "surprisingly useful" necessary condition.

Theorem 18.1. Let S be a set of vertices of a hamiltonian graph G. Then the number of connected component of $G - S$ satisfies $c(G - S) \leq |S|$. Moreover, if equality holds, then each of the $|S|$ components of $G - S$ is traceable, and every Hamilton cycle of G includes a Hamilton path in each of these components.

Definition. A graph G is *tough* if for every nonempty proper subset S of V we have $c(G - S) \leq |S|$.

Note. A graph that is not tough is not hamiltonian (this is the contrapositive of the first claim of Theorem 18.1). For example, for the graph G in Figure 18.2(a) with S as the three dark vertices, we have that $c(G-S) = 4 > 3 = |S|$ (see Figure 18.2(b)). Hence this graph is not tough and so not hamiltonian. This example may make it appear that it is easy to check a graph for hamiltonicity using Theorem 18.1, but it depends on finding set S. In fact, it was shown in B. Bauer, S. Hakimi, and E. Schmeichel "Recognizing Tough Graphs in $\mathcal{NP}$-Hard," *Discrete Applied Mathematics*, **28**, 191–195 (1990), that applying Theorem 18.1 to show a graph is not hamiltonian is $\mathcal{NP}$-hard. The paper can be viewed online on the Science Direct website (accessed 7/12/2022).

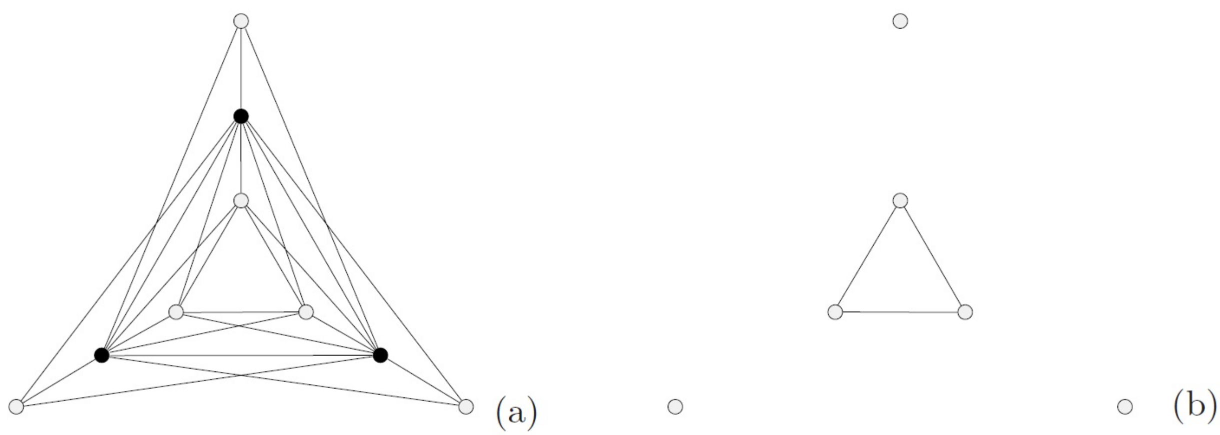

Fig. 18.2. (a) A nontough graph G, (b) the components of $G-S$

Note. The Petersen graph is nonhamiltonian (by Exercise 17.1.8), but Bondy and Murty state (see page 479) that this cannot be deduced from Theorem 18.1. In fact, the deletion of any vertex of the Petersen graph results in a hamiltonian graph (as is to be shown in Exercise 18.1.16(a)). This property defines another class of graphs.

Definition. A nonhamiltonian graph for which the deletion of any one vertex results in a hamiltonian graph is *hypohamiltonian*.

Note. The Petersen graph is vertex transitive (that is, for any two vertices there is an automorphism interchanging the two vertices; this is effectively shown in Exercise 1.2.5). Vertex-transitive hypohamiltonian graphs seem to be "extremely rare" (page 479). Another example is the Coxeter graph, given in Figure 18.3, as is to be shown in Exercises 18.1.14 and 18.1.16(c).

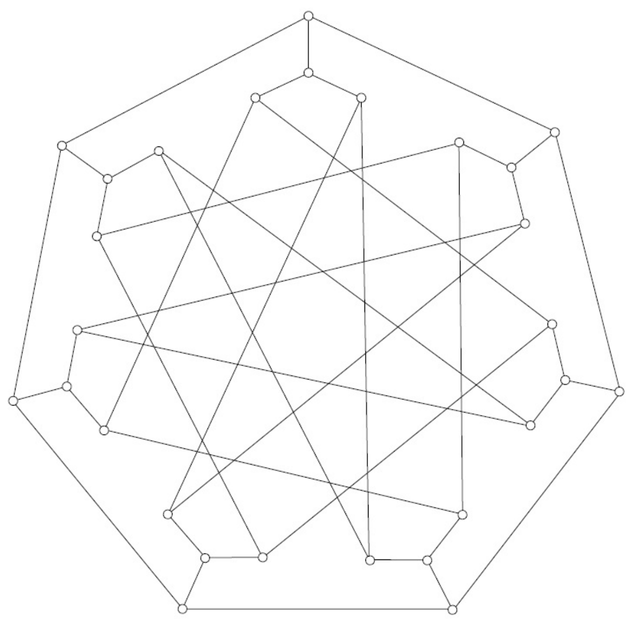

Fig. 18.3. The Coxeter graph