Chapter 2. Subgraphs

Section 2.1. Subgraphs and Supergraphs

Note. We define a subgraph of a graph, a maximal/minimal subgraph (with some given properties), and give theorems which concern the existence of a subgraph which is a cycle.

Definition. If $e \in E(G)$ then the graph H defined as $V(H) = V(G)$ and $E(H) = E(G) \setminus \{e\}$, then $H = G \setminus e$ results from G by the process of *edge deletion*. If $v \in V(G)$ then the graph K defined as $V(K) = V(G) \setminus \{v\}$ and $E(K) = \{e \in E(G) \mid e \text{ is not incident with vertex } v\}$, then $K = G - v$ results from G by the process of *vertex deletion*.

Note. Edge deletion and vertex deletion is illustrated in Figure 2.1.

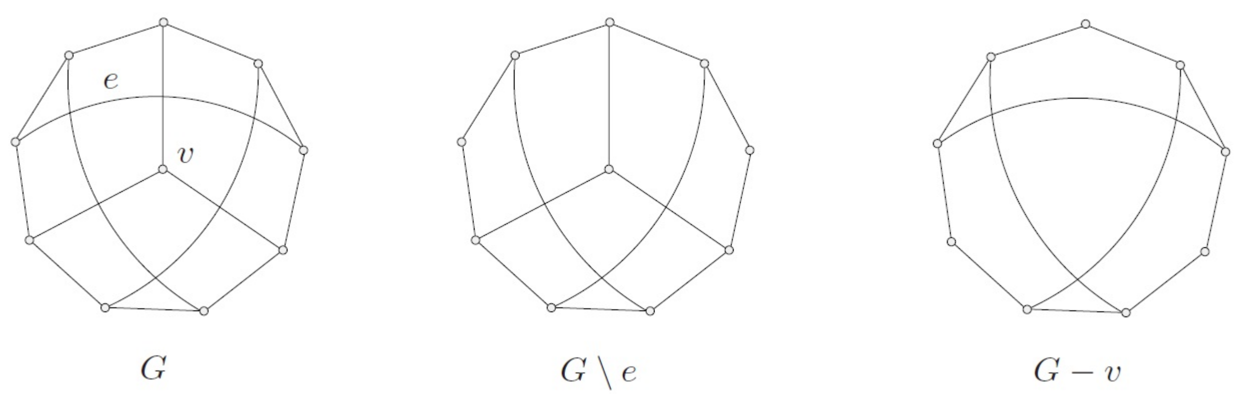

Figure 2.1. The square, triangular, and hexagonal lattices.

Definition. The graph defined above $G \setminus e$ is an *edge-deleted subgraph* of G, and the graph $G - v$ is a *vertex-deleted subgraph*. A graph F is a *subgraph* of a graph G if $V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$, and ψ_F is the restriction of ψ_G to $E(F)$. We say G *contains* F or F is *contained in* G and write $G \supseteq F$ or $F \subseteq G$, respectively.

Note. If F is a subgraph of G then there is a sequence of edge deletions and vertex deletions that converts G into F.

Definition. A *copy* of graph F in a graph G is a subgraph of G which is isomorphic to F. Such a subgraph is an F-subgraph of G. An *embedding* of a graph F in a graph G is an isomorphism between F and a subgraph of G. A *supergraph* of a graph G is a graph H which contains G as a subgraph, $H \supseteq G$. Subgraphs $F \neq G$ and supergraphs $H \neq G$ of G are called *proper*.

Note. We can extend the above definitions to digraphs and arcs in an obvious way.

Note. We will often look for subgraphs of a given graph where the subgraph has some specific property. The next theorem is our first result in this direction.

Theorem 2.1. Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.

Note. The proof of Theorem 2.1 is based on the use of a longest path. The proof is (arguably) not constructive since we do not know how to find a longest path. A "maximal" path is easy to construct (start at some vertices and extend it to a path until it cannot be further extended) and could have been used instead of a longest path. This inspires us to define maximal subgraphs.

Definition. Let $\mathcal{F}$ be a family of subgraphs of a graph G. A member F of $\mathcal{F}$ is *maximal* in $\mathcal{F}$ if no member of $\mathcal{F}$ properly contains F. A member F of $\mathcal{F}$ is *minimal* in $\mathcal{F}$ if no member of $\mathcal{F}$ is properly contained in F. When $\mathcal{F}$ is the set of all paths of G, then a maximal member of $\mathcal{F}$ is a *maximal path* of G.

Note. In Exercise 2.1.1 it is to be shown that the maximal connected subgraphs of a graph are its components. We'll see the minimal nonbipartite subgraphs of a graph are odd cycles.

Definition. In a (finite) graph G which has at least one cycle, the length of a longest cycle is the *circumference* of G and the length of a shortest cycle is the *girth* of G. A graph is *acyclic* if it does not contain a cycle. A digraph is acyclic if it does not contain a directed cycle.

Definition. A (*strict*) *partially ordered set* (or *poset*) is an ordered pair $P = (X, \prec)$ where X is a set and $\prec$ is a (*strict*) *partial order* on X. That is,

1. for no $a \in X$ do we have $a \prec a$ (i.e., $\prec$ is *irreflexive*),

2. if $a \prec b$ then we do not have $b \prec a$ (i.e., $\prec$ is *antisymmetric*), and

3. if $a \prec b$ and $b \prec c$ then $a \prec c$ (i.e., $\prec$ is *transitive*).

Two elements $u, v \in X$ are *comparable* if either $u \prec v$ or $v \prec u$ and *incomparable* otherwise. A set of pairwise comparable elements in P is a *chain*. A set of pairwise incomparable elements is an *antichain*.

Note. You might by familiar with a (nonstrict) partial ordering $\preceq$ which is reflexive, antisymmetric (in the sense that $u \preceq v$ and $v \preceq u$ implies $u = v$), and transitive. See my online notes for Complex Analysis 1 [MATH 5510] on Ordering the Complex Numbers for this and other ordering ideas. An extensive coverage of posets is given in "Graduate Combinatorics" (not an official ETSU class). See my online notes for Graduate Combinatorics (in preparation); in particular, notice "Chapter 5. Counting with Partially Ordered Sets" and "Section 5.1. Basic Properties of Partially Ordered Sets."

Definition. For a given poset $P = (X, \prec)$, define a digraph $D = D(P)$ where $V(D) = X$ and $A(D) = \{(u, v) \mid u, v \in X \text{ and } u \prec v\}$.

Note. The digraph $D(P)$ is acyclic (because of transitivity of $\prec$), and if $(u, v), (v, w) \in A(D)$ then $(u, w) \in A(D)$ (also by transitivity of $\prec$). This second property is called "transitive" for this digraph. Conversely, a strict (i.e., no loops or parallel arcs) acyclic transitive digraph D can be used to construct a poset on $X = V(D)$.

Note. We give one more result concerning the existence of a subgraph of a certain type.

Theorem 2.2. Any simple graph G with $\displaystyle\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.

Revised: 9/22/2022