Section 2.2. Spanning and Induced Subgraphs

Note. We define a spanning subgraph of a given graph, a Hamilton path and a Hamilton cycle, underlying simple graph, induced subgraph, and weighted graph. We present theorems on the existence of certain spanning and induced subgraphs, and state the Traveling Salesman Problem.

Definition. A *spanning subgraph* of a graph G is a subgraph obtained by edge deletions only (so that a spanning subgraph is a subgraph of G with the same vertex set as G). With S a set of deleted edges, the spanning subgraph is denoted $G \setminus S$. The inverse operation of edge deletion is *edge addition* where an edge is added to the edge set of G. Adding a set S of edges to a graph G yields a *spanning supergraph* of G, denoted $G + S$. With graphs G and H as disjoint graphs, the *join* of G and H, denoted $G \vee H$, is graph $G \vee H = G \cup H + S$ where S is the set of all edges joining a vertex of G to a vertex of H. The join $C_n \vee K_1$ of a cycle and a single vertex is a *wheel* with n *spokes* denoted W_n. If X is a set of vertices, we can add X to graph G to produce the supergraph denoted $G + X$.

Definition. A spanning path in a graph G is a *Hamilton path* in G. A spanning cycle in a graph G is a *Hamilton cycle* in G. A spanning k-regular subgraph of a graph G is a *k-factor* of G.

Note. Hamilton paths and cycles are named for William Rowan Hamilton (August 4, 1805–September 2, 1865), the discoverer of the quaternions. He described spanning paths and cycles in a letter to his friend John Graves. The reference for the letter is:

> W. R. Hamilton, Letter to John T. Graves "On the Icosian" (17 October 1856), *Mathematical Papers, Vol. III, Algebra*, eds. H. Halberstam and R. E. Ingram, Cambridge University Press, Cambridge (1967), 612–625.

The "icosian" is the noncommutative group of order 60 that is the group of symmetries of the regular icosahedron (or the regular dodecahedron). In fact it is isomorphic to the alternating group A_5.

(Image from The MacTutor History of Mathematics archive Hamilton biography webpage.)

The following result relates to directed Hamilton paths and is due to L. Rédie (1934).

Theorem 2.3. Rédi's Theorem. Every tournament has a directed Hamilton path.

Note. The proof of Theorem 2.3 is based on induction and can be used to create a *recursive* algorithm to actually find a Hamilton path in a tournament. A more useful algorithm would give a *direct* method of construction of the Hamilton path (though we were simply proving existence in Theorem 2.3).

Note. Though every tournament has a Hamilton path, not every tournament has a Hamilton cycle (consider the orientations of K_3, and the orientations of K_4 given in Figure 1.25 of Section 1.5).

Definition. Using edge deletion to delete from a graph G all loops and, for every pair of adjacent distinct vertices, all but one edge joining them, we create the *underlying simple graph* of G.

Definition. Given spanning subgraphs $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ of a graph $G = (V, E)$, we form the spanning subgraph of G whose edge set is the symmetric difference $E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$. This graph is the *symmetric difference* of graphs F_1 and F_2, denoted $F_1 \triangle F_2$.

Note. Figure 2.4 gives the symmetric difference of two spanning graphs on a set of five vertices.

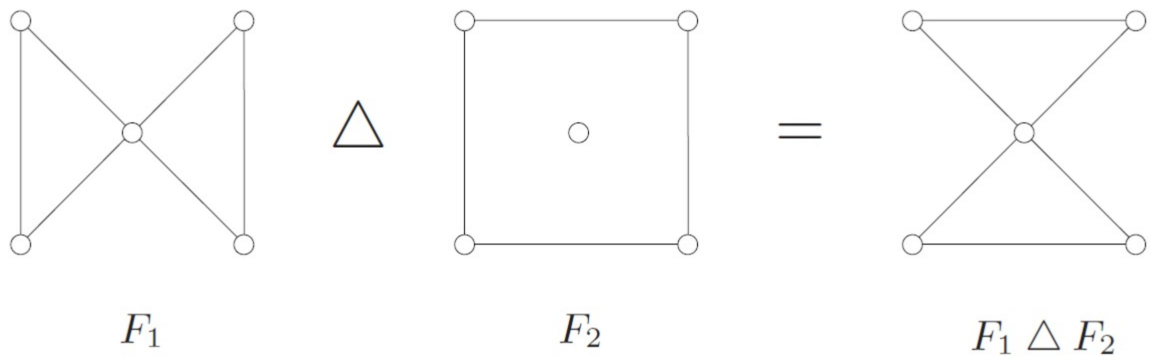


Figure 2.4

Note. In connection with spanning subgraphs we have the following result. This result is due to Paul Erdös (1965).

Theorem 2.4. Every loopless graph G contains a spanning bipartite subgraph F such that $d_F(v) \geq \frac{1}{2}d_G(v)$ for all $v \in V$.

Definition. A subgraph obtained from graph G by vertex deletion only is an *induced subgraph* of G. If X is the set of deleted vertices, the induced subgraph is denoted $G - X$. With $Y = V(G) \setminus X$, the induced subgraph is denoted $G[Y]$ and called the subgraph of G *induced by* vertex set Y.

Note. The next result is also due to Paul Erdös (1964/65) and "tells us that every graph has a induced subgraph whose minimum degree is relatively large," as Bondy and Murty say (see page 49).

Theorem 2.5. Every graph with average degree at least $2k$, where $k \in \mathbb{N}$, has an induced subgraph with minimum degree at least $k+1$.

Note. It is to be shown in Exercise 3.1.6 that the lower bound of $k+1$ on the minimum degree of an induced subgraph is best possible (or "sharp").

Definition. Let S be a set of edges of a graph G. The *edge induced subgraph* $G[S]$ is the subgraph of G whose edge set is S and whose vertex set consists of all ends of edges in S.

Note. We don't address many applications in this class, but applications inspire many graph theoretic concepts. One such concept is a weighted graph where a weight is associated with each edge of a graph and then, for example, a Hamilton path is desired that minimizes the sum of the weights of the edges in the path. The vertices could represent locations and the weights on the edges represent distances or the expense of traveling between locations; a Hamilton path would represent a way to visit every location by minimizing total distance or expense (though we might desire to start and end at particular locations or to return to our stating point).

Definition. Let G be a graph and let $w : E(G) \to \mathbb{R}$ so that each edge e of G is associated with a real number $w(e)$ called the *weight* of the edge. Graph G together with weight function w is a *weighted graph*, denoted (G, w). If F is a subgraph of G, the *weight of subgraph* F is $w(F) = \sum_{e \in E(F)} w(e)$.

Note. Many discrete optimization problems can be described in terms of weighted graphs. Probably the most famous is the Traveling Salesman Problem. Consider a salesman that needs to visit a number of towns and return to the starting point. The salesman wishes to do so in a way that minimizes time traveled, distance traveled, the expense of travel, etc. What should be the salesman's travel itinerary in order to obtain a minimization? This is the Traveling Salesman Problem. Stated in graph theoretic terms, it is the following.

Problem 2.6. THE TRAVELING SALESMAN PROBLEM (TSP).
Given a weighted complete graph (G, w), find a minimum-weight Hamilton cycle of G.

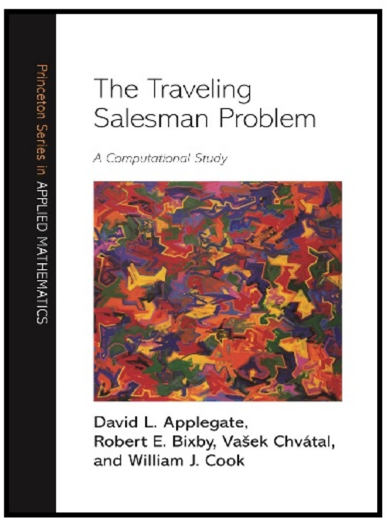
Note. We consider complete graphs without loss of generality, since we could assign relatively large weights to specific edges so that they cannot be in a minimum-weight Hamilton cycle, effectively removing those edges from the graph (of course, we cannot remove too many edges and still be insured that a Hamilton cycle exists).

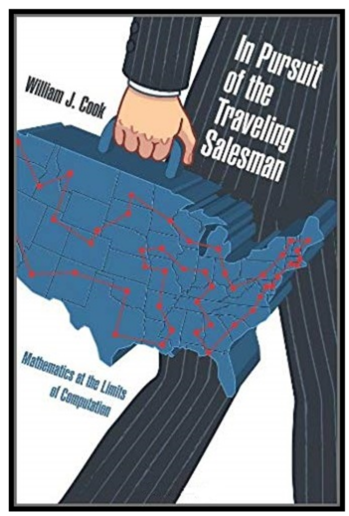
Note. Since we consider finite complete graphs, a minimum-weight Hamilton cycle of G necessarily exists. The problem is how to *find* a minimum-weight solution. This and related ideas are explored more in Chapter 6 (Tree Search Algorithms) and Chapter 8 (Complexity of Algorithms).

Note. A couple of references on the Traveling Salesman Problem are:

- *The Traveling Salesman Problem: A Computational Study* (Princeton Series in Applied Mathematics) by David Applegate, Robert Bixby, Vašek Chvátal, and William Cook, Princeton University Press (2006).
- *In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation* by William Cook, Princeton University Press (2012).

The first of these is somewhat technical, but includes a good deal of history of the problem and is written by active researches on the TSP. The second of these is written at a more popular level.





The ETSU Sherrod Library has a copy of *The Traveling Salesman Problem: A Computational Study* (QA164.T72 2006); the book is also available for online reading through the library. The ETSU Sherrod Library also has a copy of *In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation* (QA164.C69 2012) and it too can be read online (to read books online, you have to sign in using your ETSU credentials).

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