Section 2.3. Modifying Graphs

Note. In this brief section we describe ways to create new graphs from given graphs. Unlike with edge-deletion and vertex deletion, these techniques produce new graphs that are not (necessarily) subgraphs of the given graph.

Definition. Let x and y be two nonadjacent vertices of graph G. If we replace x and y with a single vertex v and replace all edges of G incident to either x or y with edges incident to v with the other end of such edges the same as they were in G, then we *identify* x and y to produce a new graph $G/\{x, y\}$. To *contract* edge e of G, we delete e from G and (if e is a link) identify the end vertices of e, producing a new graph G/e.

Note. Figure 2.5 illustrates vertex identification and edge contraction.

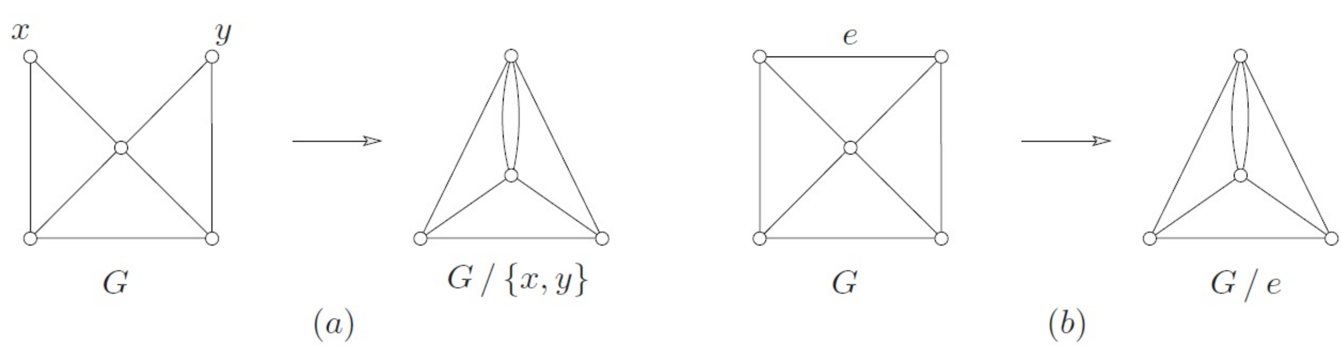

Figure 2.5. (a) Identifying two vertices, and (b) contracting an edge.

You might recognize the notation used here as similar to that used in quotient groups (or "factor groups") where cosets are collections of group elements "identified" together using a normal subgroup. For example,

$$\mathbb{Z}/2\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}/\{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

has elements $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ and $2\mathbb{Z} + 1 = \{\ldots, -3, -1, 0, 1, 3, \ldots\}$. The quotient group $\mathbb{Z}/2\mathbb{Z}$ (read "$\mathbb{Z}$ modulo $2\mathbb{Z}$") is isomorphic to $\mathbb{Z}_2$ (for more details, see my online notes for Introduction to Modern Algebra [MATH 4127/5127] on III.14. Factor Groups, and for Modern Algebra 1 [MATH 5410] on I.5. Normality, Quotient Groups, and Homomorphisms). For this reason, we would read $G/\{x, y\}$ as "G modulo vertices x and y," and we would read G/e as "G modulo edge e."

Note. We now give an "inverse" operation (of sorts) to edge contraction.

Definition. Let v be a vertex of graph G. If we replace vertex v by two adjacent vertices v' and v'' and replace each edge of G incident to v by an edge incident to either v' or v'' (but not both, unless the edge of G is a loop) with the other end of such edges the same as they were in G, then we *split* vertex v. To *subdivide* an edge e of G is to delete edge e, add a new vertex x, and join x to the ends of deleted edge e.

Note. Given the either/or nature of the definition of splitting a vertex, a graph that results from splitting a vertex G is not in general uniquely determined. Figure 2.26 illustrates splitting a vertex and subdividing edges.

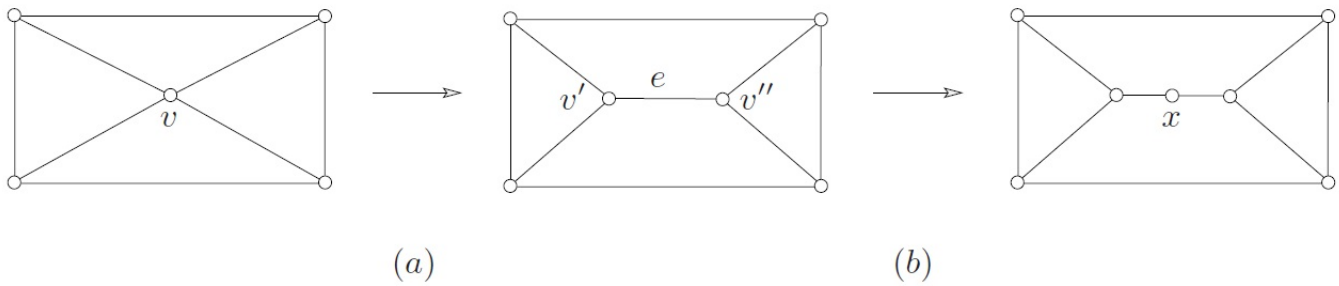

Figure 2.6. (a) Splitting a vertex, and (b) subdividing an edge.

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