# **Section 2.4.** Decompositions and Coverings

**Note.** We define graph decompositions and consider decompositions involving cycles and complete bipartite graphs. We define graph coverings and allude to graph packings.

**Definition.** A *decomposition* of a graph $G$ is a family $\mathcal{F}$ of edge-disjoint subgraphs of $G$ such that $\cup_{F \in \mathcal{F}} E(F) = E(G)$. If the family $\mathcal{F}$ consists entirely of paths or entirely of cycles, we call $\mathcal{F}$ a *path decomposition* or *cycle decomposition* of $G$, respectively. If every subgraph of $G$ in $\mathcal{F}$ is isomorphic to some particular subgraph $H$ of $G$, then we call $\mathcal{F}$ an *isomorphic decomposition* of $G$.

**Definition.** A graph is *even* if each of its vertices has even degree.

**Note.** Every vertex of a cycle is of even degree, so a necessary condition for a cycle decomposition of graph $G$ is that each vertex of $G$ is of even degree; that is, $G$ is an even graph. In fact, this is also sufficient as shown by O. Veblen (1912/13) in the following.

**Theorem 2.7.** VEBLEN'S THEOREM.
A graph admits a cycle decomposition if and only if it is even.

**Note.** A directed version of Veblen's Theorem is to be proved in Exercise 2.4.2:

> A directed graph $D$ admits a decomposition into directed cycles if and only if $d^-(v) = d^+(v)$ for all $v \in V$. (A digraph satisfying this in-degree/out-degree condition is called an *even* digraph.)

**Note.** The next result concerns decompositions of $K_n$ into complete bipartite graphs. It puts a lower bound on the number of needed complete bipartite graphs for the decomposition. The result is due to Ron Graham and H. O. Pollack (1971), but the proof presented in the book is due to H. Tverberg (1982). It is based on systems of homogeneous equations (which, I think, somewhat masks the underlying graph theoretic ideas).

**Theorem 2.8.** Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a decomposition of $K_n$ into complete bipartite graphs. Then $k \geq n - 1$.

**Definition.** A *covering* or *cover* of a graph $G$ is a family $\mathcal{F}$ of subgraphs of $G$, not necessarily edge-disjoint, such that $\cup_{F \in \mathcal{F}} E(F) = E(G)$. The *number of times edge $e$ is covered* by the covering is the number of times that $e$ appears in $\cup_{F \in \mathcal{F}} E(F)$ when treating this as a multiset union (which allows repetition of elements). A covering is *uniform* if it covers each edge of $G$ the same number of times; when this number of times is $k$ the covering is called a *$k$-cover*. A 2-cover is often called a *double cover*. If the family $\mathcal{F}$ consists only of paths or only of cycles then the covering is called a *path covering* or *cycle covering*, respectively.

**Note.** A 1-cover of a graph is simply a decomposition of the graph. The topic of Section 3.5 is cycle double covers of graphs. Chapter 19 is titled "Coverings and Packings in Directed Graphs," so we have not finished with these ideas!

**Note.** If $\mathcal{F}$ is a covering of a simple graph $G$ but not a decomposition, then some of the elements of the $F_i$s must share edges and some of the edges of $G$ must be "covered" more than once. In this case, treating edge sets as multi-sets (in which elements can appear multiple times) we have a quantity $|\cup_{F_i \in \mathcal{F}} E(F_i) \setminus E(G)|$ which reflects the "extra edges" in the covering; this collection of edges is called the *padding* of the covering. It is a topic of interest to minimize the size of the padding over particular types of coverings (yielding a "minimal covering"). Graph decompositions, graph coverings, and the related topic of graph packings are explored in a supplement to this section (at least in the particular case where $F_i \cong F_j$ for all $F_i, F_j \in \mathcal{F}$; that is, for isomorphic decompositions/coverings/packings).

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