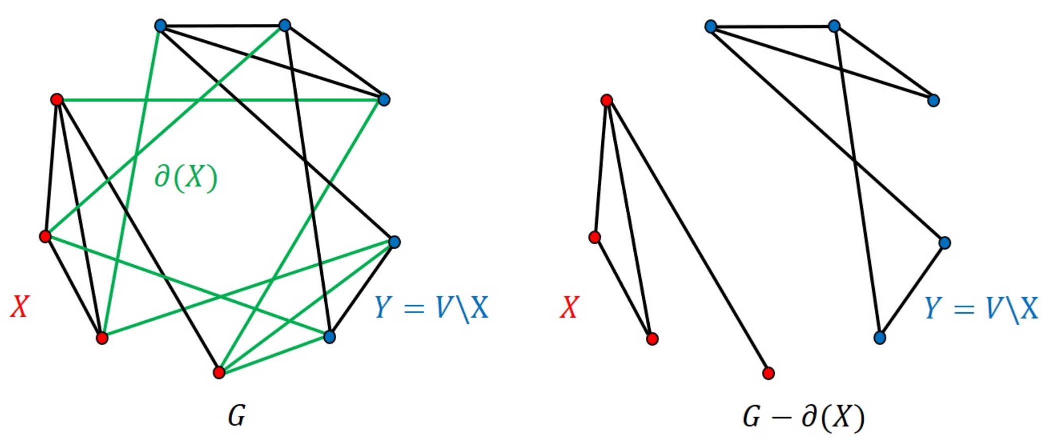
Section 2.5. Edge Cuts and Bonds

Note. In this section we introduce an operation that, based on a given set of vertices, deletes several edges at a time from a graph. We give several properties of this operation, called "edge cut." In the previous sections we have stated several definitions and presented a few results. With this foundation, we now start to give more theorems and corollaries per section.

Definition. Let X and Y be sets of vertices (not necessarily disjoint) of a graph $G = (V, E)$. Denote by $E[X, Y]$ the set of edges of G with one end in X and the other end in Y, and denote by $e(X, Y)$ their number. If $Y = X$, we write $E(X, Y) = E(X, X) = E(X)$ and $e(X, Y) = e(X, X) = e(X)$. When $Y = V \setminus X$, the set $E[X, Y] = E[X, V \setminus X]$ is the *edge cut* of G associated with X, or the *coboundary* of set X, denoted $\partial(X)$. An edge cut $\partial(v)$ associated with a single vertex v is a *trivial edge cut*. In a loopless graph, $|\partial(X)|$ is the *degree* of set X, denoted $d(X)$.

Note. The term "edge cut" is used above because removing $\partial(X)$ from graph G results in a disconnected graph:



Note 2.5.A. With $Y = V \setminus X$, $\partial(X) = \partial(Y)$. Also $\partial(V) = \varnothing$. If for $G = (V, E)$ we have $\partial(X) = E$ for some $X \subset V$ then G is bipartite with bipartition $(X, V \setminus X)$. Graph $G(V, E)$ is connected if $\partial(X) \neq \varnothing$ for every nonempty proper subset X of V. Figure 2.8 gives several examples of edge cuts. Notice that $V = \{u, v, x, y\}$ has $2^4 = 16$ subsets, but Figure 2.8 only includes 8 edge cuts. This is because $\partial(X) = \partial(V \setminus X)$.

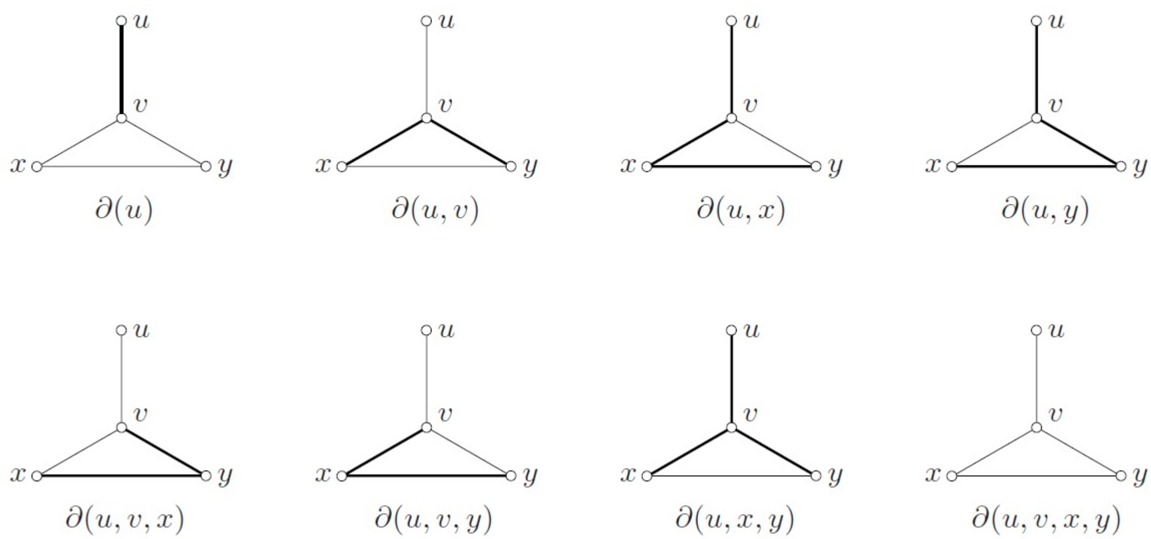


Figure 2.8

Theorem 2.9. For any graph G and any subset X of V, we have

$$|\partial(X)| = \sum_{v \in X} d(v) - 2e(X).$$

Note. Theorem 2.9 is a generalization of Theorem 1.1, since with $X = V$ we have $|\partial(X)| = |\partial(V)| = |\varnothing| = 0$ and $e(X) = e(V) = m$. A proof of Theorem 2.9 is to be given in Exercise 2.5.1(a).

Note. Veblen's Theorem (Theorem 2.7) gives a classification of even graphs in terms of cycle decompositions. The next theorem also classifies even graphs in terms of edges cuts.

Theorem 2.10. A graph $G = (V, E)$ is even if and only if $|\partial(X)|$ is even for every subset X of V.

Note. We now consider the interaction of symmetric differences and edge cuts.

Proposition 2.11. Let G be a graph and let X and Y be subsets of V. Then $\partial(X)\triangle\partial(Y) = \partial(X\triangle Y)$.

Note. The following corollary is really just a restatement of Proposition 2.11.

Corollary 2.12. In a given graph G, the symmetric difference of two edge cuts is itself an edge cut.

Note. The following result relates the edge cut of a set of vertices X in a symmetric difference of two graphs to the symmetric difference of the edge cuts of X (all this occurring in a given graph G). The proof is to be given in Exercise 2.5.1(b).

Proposition 2.13. Let F_1 and F_2 be spanning subgraphs of a graph G, and let $X \subset V$. Then

$$\partial_{F_1 \triangle F_2}(X) = \partial_{F_1}(X) \triangle \partial_{F_2}(X).$$

Definition. A *bond* of a graph is a minimal nonempty edge cut; that is, a nonempty edge cut none of whose nonempty proper subsets (of edges) is an edge cut.

Note. We can exhaustively check that the bonds of the graph with the given edge cuts in Figure 2.8 are the edge cuts given in Figure 2.11. For example, $\partial(u, v) = \{vx, vy\}$ is a bond since there is no edge cut of the form $\{vx\}$ or $\{vy\}$.

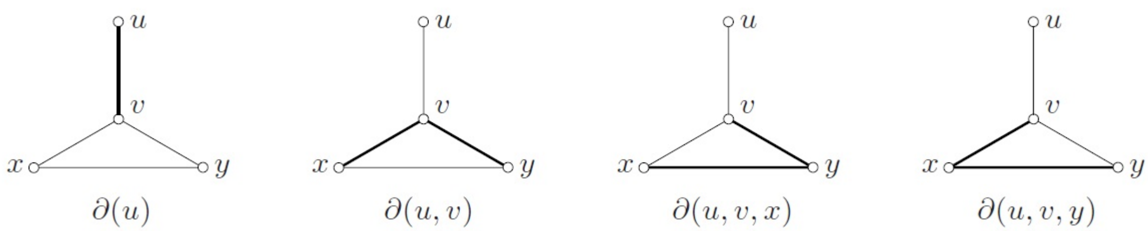


Figure 2.11. The bonds of the graph with edge cuts as given in Figure 2.8.

Note. In Figure 2.8 (see again below), none of the edge cuts $\partial(u, x)$, $\partial(u, y)$, $\partial(u, x, y)$ are bonds since each contains edge uv. Edge cut $\partial(u, v, x, y)$ is not a bond (by definition) since it is empty.

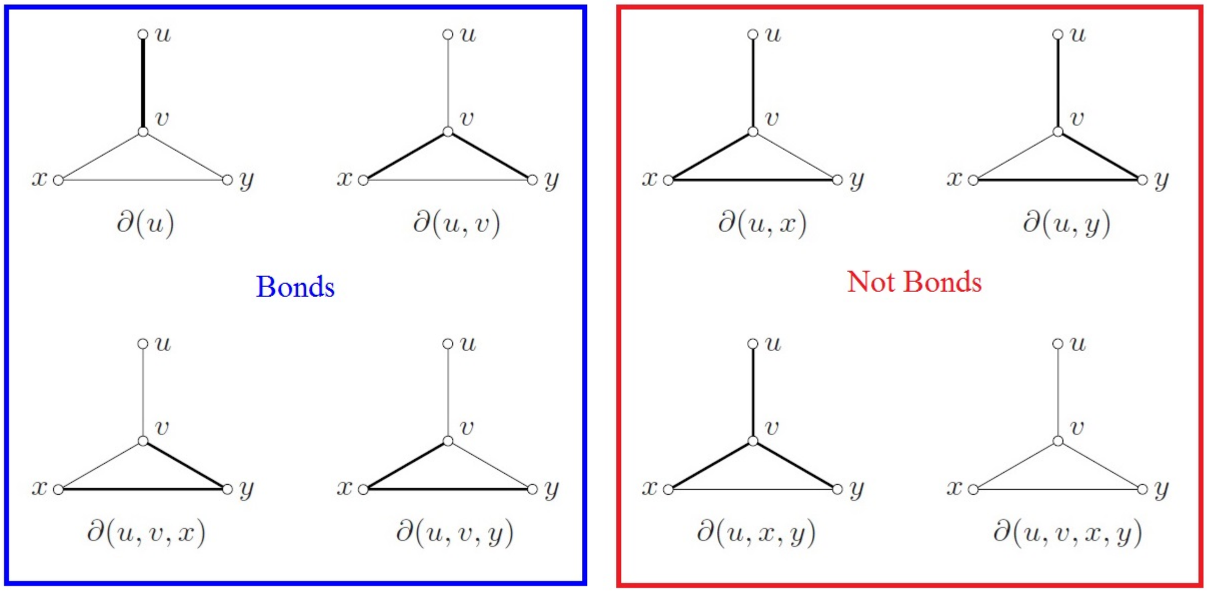


Figure 2.8′. Bonds and non-bonds.

Note. The following two results relate edge cuts and bonds. Theorem 2.14 (the proof of which is to be given in Exercise 2.5.1(c)) shows that bonds sort of "make up" edge cuts (similar to how a set of basis vectors "make up" a vector space).

Theorem 2.14. A set of edges of a graph is an edge cut $\partial(X)$ if and only if it is a disjoint union of bonds.

Theorem 2.15. In a connected graph G, a nonempty edge cut $\partial(X)$ is a bond if and only if both $G[X]$ and $G[V \setminus X]$ are connected.

Note. Notice above that the bonds $\partial(u)$, $\partial(u, v)$, $\partial(u, v, x)$, $\partial(u, v, y)$ in Figure 2.18′ and the non-bonds $\partial(u, x)$, $\partial(u, y)$, $\partial(u, x, y)$, $\partial(u, v, x, y)$ all satisfy Theorem 2.15.

Note. We now extend the idea of edge cut to directed graphs.

Definition. Let X and Y be sets of vertices (not necessarily disjoint) of a digraph $D = (V, A)$. Denote by $A(X, Y)$ the set of arcs of D whose tails lie in X and whose heads lie in Y, and denote by $a(X, Y)$ their number. If $X = Y$, we write $A(X, Y) = A(X, X) = A(X)$ and $a(X, Y) = a(X, X) = a(X)$. When $Y = V \setminus X$, the set $A(X, Y) = A(X, V \setminus X)$ is the *outcut* of D associated with X, denoted $\partial^+(X)$. Similarly, the set $A(X, Y) = A(V \setminus X, X)$ is the *incut* of D associated with X, denoted $\partial^-(X)$. In a loopless digraph, $|\partial^+(X)|$ and $|\partial^-(X)|$ are the *outdegree* and *indegree* of set X denoted $d^+(X)$ and $d^-(X)$, respectively. A digraph D is *strongly connected* or *strong* if $\partial^+(X) \neq \varnothing$ for every nonempty proper subset X of V.

Note. In $D = (V, A)$ we have $\partial^+(X) = \partial^-(V \setminus X)$. We denote $\partial(X) = \partial^+(X) \cup \partial^-(X)$ for each $X \subseteq V$. So D is strongly connected if $\partial^-(X) \neq \varnothing$ for every nonempty proper subset X of V.

Definition. A *directed bond* in digraph $D = (V, A)$ is a bond $\partial(X) = \partial^+(X) \cup \partial^-(X)$ such that $\partial^-(X) = \varnothing$. That is, $\partial(X) = \partial^+(X) \cup \partial^-(X) = \partial^+(X)$.

Note. Directed bonds are addressed in Exercise 2.5.7 where it is to be shown (among other results) that a digraph is acyclic if and only if every bond is a directed bond.

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