Section 2.6. Even Subgraphs

Note. Recall from Section 2.4, "Decompositions and Coverings," that an even graph is a graph in which every vertex has even degree. In this section we consider the symmetric difference of even subgraphs of a given graph and we see how even subgraphs interact with edge cuts. We also define three vector spaces associated with a graph, the edge space, the cycle space, and the bond space.

Definition. For a given graph G, an *even subgraph* of G is a spanning subgraph of G which is even. (Sometimes we may use the term "even subgraph of G" to indicate just the edge set of the subgraph.)

Note. Theorem 2.10 and Proposition 2.13 of the previous section allow us to address the symmetric difference of two even subgraphs of a given graph, as follows.

Corollary 2.16. The symmetric difference of two even subgraphs is an even subgraph.

Note. Bondy and Murty comment that "…we show in Chapters 4 ["Trees"] and 21 ["Integer Flows and Coverings"], [that] the even subgraphs of a graph play an important structural role." See page 64. In the context of even subgraphs, the term "cycle" may be used to mean the edge set of a cycle, and the term "disjoint cycles" may mean edge-disjoint cycles. With this convention, Veblen's Theorem (Theorem 2.7), which states that a graph admits a cycle decomposition if and only if it is even, can be restated as follows.

Theorem 2.17. A set of edges of a graph is an even subgraph if and only if it is a disjoint union of cycles.

Note. A quick cautionary comment: In Introduction to Modern Algebra (MATH 4127/5125) is is shown that "Every permutation of a finite set is a product of disjoint cycles." See Theorem 9.8 in my online notes on II.9. Orbits, Cycles, and the Alternating Groups. The similar wording of this result and our Theorem 2.17 is purely coincidental!

Note. The next result addresses the interaction of even subgraphs and edge cuts. This result will be useful below when we define vector spaces associated with a graph.

Proposition 2.18. In any graph, every (edge set of an) even subgraph meets every edge cut in an even number of edges.

Note. Denote the set of all subsets of the edge set E of a graph G as $\mathcal{E}(G)$; that is, $\mathcal{E}(G)$ is the power set of E, $\mathcal{E}(G) = \mathcal{P}(E(G))$. We denote the field of order 2 as $GF(2)$ (for "Galois field of order 2"), and of course $GF(2) \cong \mathbb{Z}_2$ (here $\cong$ represents a field isomorphism). We denote the elements of $GF(2)$ as 0 and 1.

Definition. For a graph G, the vector space formed on the set of vectors $\mathcal{E}(G)$ with scalar field $GF(2)$, where for $E_1, E_2 \in \mathcal{E}(G)$ we define vector addition as $E_1 + E_2 = E_1 \triangle E_2$ and scalar multiplication is defined as $0E_1 = \varnothing$ and $1E_1 = E_1$, is the *edge space* of graph G, also denoted $\mathcal{E}(G)$.

Note. In Exercise 2.6.2 it is to be shown that the edge space of a graph actually is a vector space over $GF(2)$. Notice that if sets of edges $E_1, E_2 \in \mathcal{E}(G)$ are disjoint, then $E_1 + E_2 = E_1 \triangle E_2 = E_1 \cup E_2$, so when $E(G) = \{e_1, e_2, \ldots, e_m\}$ then a basis (called the *standard basis*) for the edge space is given by $\{e_1\}, \{e_2\}, \ldots, \{e_m\}$ so that $\dim(\mathcal{E}(G)) = m$. We can then write any $E_1 \in \mathcal{E}(G)$ as $E_1 = \sum_{i=1}^m a_i\{e_i\}$ where $a_i \in GF(2)$ for each $i \in \{1, 2, \ldots, m\}$. We can now use this standard basis to put an inner product on $\mathcal{E}(G)$.

Definition. Let $E_1, E_2 \in \mathcal{E}(G)$. Suppose $E_1 = \sum_{i=1}^m a_i\{e_i\}$ and $E_2 = \sum_{i=1}^m b_i\{e_i\}$ where $\{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$ is the standard basis of $\mathcal{E}(G)$. The *inner product* of E_1 and E_2 is

$$\langle E_1, E_2 \rangle = \left\langle \sum_{i=1}^m a_i\{e_i\}, \sum_{i=1}^m b_i\{e_i\} \right\rangle = \sum_{i=1}^m a_i b_i.$$

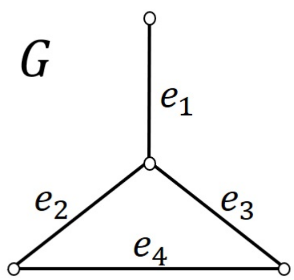
If $\langle E_1, E_2 \rangle = 0$ then E_1 and E_2 are *orthogonal*, denoted $E_1 \perp E_2$.

Note 2.6.A. For $E_1, E_2 \in \mathcal{E}(G)$, we have $E_1 \perp E_2$ if and only if $|E_1 \cap E_2|$ is even (in which case $\langle E_1, E_2 \rangle$ consists of a sum of an even number of 1's).

Definition. For a graph $G = (V, E)$, with each subset X of E (i.e., $X \in \mathcal{E}(G)$), define the *incidence vector* of X, denoted $\mathbf{f}_X$, as having $|E|$ components given by $f_X(e) = 1$ if $e \in X$ and $f_X(e) = 0$ if $e \notin X$ (so $\mathbf{f}_X$ is a vector with $|E|$ components and each component is in $GF(2)$). Then $\mathbf{f}_X \in GF(2)^{|E|}$. (We denote the vector space of dimension $|E|$ with scalar field $GF(2)$ simply as $GF(2)^{|E|} = GF(2)^E$.)

Note. Of course we have that $E_1, E_2 \in \mathcal{E}(G)$ satisfy $\langle E_1, E_2 \rangle = 0$ if and only if $\langle \mathbf{f}_{E_1}, \mathbf{f}_{E_2} \rangle = 0$.

Note 2.6.B. Consider graph G with edges e_1, e_2, e_3, e_4 as given here:



Then the power set of the edge set, $\mathcal{P}(E) = \mathcal{E}(G)$, is

$$\mathcal{E}(G) = \{\varnothing, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\},$$

$$\{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\}.$$

Notice that $|\mathcal{E}(G)| = 2^4 = 16$. The incidence vectors are then:

$$\mathbf{f}_\varnothing = [0,0,0,0], \mathbf{f}_{\{e_1\}} = [1,0,0,0], \mathbf{f}_{\{e_2\}} = [0,1,0,0], \mathbf{f}_{\{e_3\}} = [0,0,1,0],$$

$$\mathbf{f}_{\{e_4\}} = [0,0,0,1], \mathbf{f}_{\{e_1,e_2\}} = [1,1,0,0], \mathbf{f}_{\{e_1,e_3\}} = [1,0,1,0], \mathbf{f}_{\{e_1,e_4\}} = [1,0,0,1],$$

$$\mathbf{f}_{\{e_2,e_3\}} = [0,1,1,0], \mathbf{f}_{\{e_2,e_4\}} = [0,1,0,1], \mathbf{f}_{\{e_3,e_4\}} = [0,0,1,1],$$

$$\mathbf{f}_{\{e_1,e_2,e_3\}} = [1,1,1,0], \mathbf{f}_{\{e_1,e_2,e_4\}} = [1,1,0,1], \mathbf{f}_{\{e_1,e_3,e_4\}} = [1,0,1,1],$$

$$\mathbf{f}_{\{e_2,e_3,e_4\}} = [0,1,1,1], \mathbf{f}_{\{e_1,e_2,e_3,e_4\}} = [1,1,1,1].$$

So $\{\mathbf{f}_X \mid X \in \mathcal{E}(G)\}$ consists of all 4-tuples of 0's and 1's. Together, these form the vector space $GF(2)^4$ (the 4-dimensional vector space with scalars from $GF(2)$).

Note. In Exercise 2.6.2, it is to be shown that the edge space of graph G is isomorphic to the vector space $GF(2)^E$ where E is the edge set of G.

Note 2.6.C. The only even (spanning) subgraphs of the graph G given in Note 2.6.B are the graph with edge set $\varnothing$ and the graph with edge set $\{e_2, e_3, e_4\}$. These correspond to the vectors $\mathbf{f}_\varnothing = [0,0,0,0]$ and $\mathbf{f}_{\{e_2,e_3,e_4\}} = [0,1,1,1]$. So the subspace of $\mathcal{E}(G)$ formed by the even subgraphs of G is isomorphic to $\{[0,0,0,0],[0,1,1,1]\}$.

Theorem 2.6.A. Let G be a graph. The set of all even subgraphs of G form a subspace of the edge space of G.

Note 2.6.C. The edge cuts of the graph G given in Note 2.6.B (as we saw in Figure 2.8 of Section 2.5) are $\{e_1\}$, $\{e_2, e_3\}$, $\{e_1, e_2, e_4\}$, $\{e_1, e_3, e_4\}$, $\{e_3, e_4\}$, $\{e_2, e_4\}$, $\{e_1, e_2, e_3\}$, and $\varnothing$. These correspond to the incidence vectors

$$[1,0,0,0], [0,1,1,0], [1,1,0,1], [1,0,1,1], [0,0,1,1], [0,1,0,1], [1,1,1,0], [0,0,0,0],$$

respectively.

Note. We can easily modify the proof of Theorem 2.6.A by replacing the use of Corollary 2.16 with Corollary 2.12 to get the following.

Theorem 2.6.B. Let G be a graph. The set of all edge cuts of G form a subspace of the edge space of G.

Definition. For graph G, the vector space of all (edge sets of) even subgraphs of G given in Theorem 2.6.A is the *cycle space* of G, denoted $\mathcal{C}(G)$. The vector space of all edge cuts of G given in Theorem 2.6.B is the *bond space* of G, denoted $\mathcal{B}(G)$. The dimension of the cycle space $\mathcal{C}(G)$ is the *cyclomatic number* of G.

Note. The cycle space $\mathcal{C}(G)$ of graph G (given in Note 2.6.B) consists of the two vectors given in Note 2.6.C. Notice that this space is generated by the (edge set of) the cycle in G, $\{e_2, e_3, e_4\}$. To more clearly see this, consider $\mathbf{f}_{\{e_2,e_3,e_4\}} = [0, 1, 1, 1]$ and consider $\text{span}(\{[0, 1, 1, 1]\})$ in $GF(2)^4$. So $\dim(\mathcal{C}(G)) = 1$.

Note. The bond space $\mathcal{B}(G)$ of graph G (given in Note 2.6.B) consists of the eight vectors given in Note 2.6.C. Notice that this space is generated by the bonds of G, which are given in Figure 2.8′ of Section 2.5: $\{e_1\}$, $\{e_2, e_3\}$, $\{e_3, e_4\}$, and $\{e_2, e_4\}$. To more clearly see this, consider $\mathbf{f}_{\{e_1\}} = [1, 0, 0, 0], \mathbf{f}_{\{e_2,e_3\}} = [0, 1, 1, 0], \mathbf{f}_{\{e_3,e_4\}} = [0, 0, 1, 1], \mathbf{f}_{\{e_2,e_4\}} = [0, 1, 0, 1]$ and $\text{span}(\{[1, 0, 0, 0], [0, 1, 1, 0], [0, 0, 1, 1], [0, 1, 0, 1]\})$ in $GF(2)^4$. Notice that these vectors are not linearly independent since $[0, 1, 1, 0] + [0, 0, 1, 1] = [0, 1, 0, 1]$; in fact, $\dim(\mathcal{B}(G)) = 3$.

Note 2.6.D. In Exercise 2.6.4(a) it is to be shown that the cycle space of graph G (defined in terms of even subgraphs of G) is generated by the cycles of G. In Exercise 2.6.4(b) it is to be shown that the bond space of graph G (defined in terms of edge cuts of G) is generated by the bonds of G; thus, the names of these spaces! In Exercise 2.6.4(c) it is to be shown that the cycle space and the bond space are orthogonal complements of each other in the edge space of G; this is to be accomplished by showing that the bond space of graph G is isomorphic to the row space of its incidence matrix $\mathbf{M}$ over $GF(2)$, and the cycle space of G is isomorphic to its orthogonal complement (so that the cycle space is isomorphic to the null space of $\mathbf{M}$).

Note. For the graph G given in Note 2.6.B, we have that the cycle space satisfies $\mathcal{C}(G) \cong \text{span}(\{[0,1,1,1]\})$ and the bond space satisfies

$$\mathcal{B}(G) \cong \text{span}(\{[1,0,0,0],[0,1,1,0],[0,0,1,1],[0,1,0,1]\}).$$

Notice that $[0,1,1,1]$ is orthogonal to each vector in the spanning set for (an isomorphic image of) $\mathcal{B}(G)$. Notice that $\dim(\mathcal{C}(G))+\dim(\mathcal{B}(G)) = 1+3 = 4 = \dim(\mathcal{E}(G))$. So $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are orthogonal complements in $\mathcal{E}(G)$. That is, $\mathcal{C}(G)^\perp = \mathcal{B}(G)$ and $\mathcal{E}(G) = \mathcal{C}(G) \oplus \mathcal{B}(G)$.

Note. We see in the previous note that the set of bonds form a generating set (i.e., a spanning set) for the bond space but that it may not form a basis (and the same holds for the set of cycles and the cycle space). In section 4.3 we will define a "fundamental cycle" and a "fundamental bond" with respect to a given spanning tree of a connected graph. In Exercise 4.3.6(a) it is to be shown that, for a given spanning tree T of a connected graph, the fundamental cycles with respect to T form a basis of the cycle space and the fundamental bonds with respect to T form a basis of the bond space.

Note. In fact we can describe the dimensions of the cycle space and bond space of a given connected graph in terms of the number of edges m and the number of vertices n of a graph G as follows:

$$\dim(\mathcal{C}(G)) = m - n + 1 \text{ and } \dim(\mathcal{B}(G)) = n - 1.$$

This is Bondy and Murty's Exercise 4.3.6(b) and is stated as Theorem 1.9.6 and proved in Reinhard Diestel's *Graph Theory*, Graduate Texts in Mathematics #173, (Springer, 1997); see page 22. Notice this is consistent with our example above where $n = 4$ and $m = 4$. Of course the dimension of the edges space is $|E| = m$, as observed above.

Note. In Chapter 20, "Electrical Networks," we consider edge spaces, cycle spaces, and bond spaces over arbitrary scalar fields.

Note 2.6.E. Some insight on how to extend the idea of the edge space to an arbitrary field is given in Norman Biggs' *Algebraic Graph Theory*, 2nd Edition, Cambridge University Press (1993). In Chapter 4, Cycles and Cuts, a vector space over the field $\mathbb{C}$ of complex numbers is defined in terms of a finite set X. The set of vectors is taken to be all functions $f : X \to \mathbb{C}$. Then for two such functions f

and g and for scalar $a \in \mathbb{C}$ we define

$$(f+g)(x) = f(x) + g(x) \text{ and } (af)(x) = af(x) \text{ for all } x \in X.$$

The dimension of this vector space is then the finite number $|X|$ and so, since the scalar field is $\mathbb{C}$, the vector space is isomorphic to $\mathbb{C}^{|X|}$ (from the Fundamental Theorem of Finite Dimensional Vector Spaces; see Theorem 5.1.2 of my online notes for Fundamentals of Functional Analysis [MATH 5740] on 5.1. Groups, Fields, and Vector Spaces). For graph G, the edge space is then the vector space of all functions from the edge set of G, $E(G)$, to $\mathbb{C}$; the vertex space is similarly defined. See page 23 of Biggs. Notice that this idea extends easily to functions mapping sets to an arbitrary field $\mathbb{F}$ and that this results in an edge space isomorphic to $\mathbb{F}^{|E|}$. In particular, with $\mathbb{F} = GF(2)$ we see that we get (up to isomorphism, at least) the same idea as introduced above. In this setting, for edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ and vector $f : E(G) \to GF(2)$ we have the association of f with the vector $[f(e_1), f(e_2), \ldots, f(e_m)] \in GF(2)^E$ (notice that this vector is in fact the incidence vector $\mathbf{f}_X$ for the set $X = \{e_i \mid f(e_i) = 1\}$).

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