Section 2.7. Graph Reconstruction

Note. In this section, we consider the problem of determining a graph G from the collection of $v(G)$ subgraphs of G of the form $G - v$ (called the reconstruction problem). We consider the similar problem of determining a graph G from the collection of $e(G)$ subgraphs of G of the form $G - e$ (called the edge reconstruction problem). We give classes of graphs for which these reconstruction problems are solved, and parameters which can be determined from the vertex-deleted or edge-deleted subgraphs of G. We introduce Möbius inversion and apply it to some edge-reconstruction questions.

Definition. Graphs G and H on the same vertex set V are *hypomorphic* if, for all $v \in V$, their vertex-deleted subgraphs $G - v$ and $H - v$ are isomorphic.

Note. With $G = 2K_1$ (that is, G has two vertices of degree 0) and $H = K_2$, then for vertices v_1 and v_2 (say) we have $G - v_i \cong H - v_i$ for $i = 1, 2$ so that G and H are hypomorphic by G and H are not isomorphic. So hypomorphic does not imply isomorphic. "However, these two graphs are the only known nonisomorphic pair of hypomorphic simple graphs" (Bondy and Murty, page 66). P. H. Kelly conjectured in his 1942 Ph.D. dissertation at the University of Wisconsin that there are no other such pairs (S. M. Ulam published the conjecture in *A collection of Mathematical Problems*, Interscience Press, 1960).

Definition. A *reconstruction* of a graph G is any graph hypomorphic to G. Graph G is *reconstructible* if every reconstruction of G is isomorphic to G; that is, G is reconstructible if it can be determined up to isomorphism from its vertex-deleted subgraphs.

Note. If a graph has n vertices then it has n vertex-deleted subgraphs (though some may be isomorphic to others). Figure 2.12 gives six vertex-deleted subgraphs of an unknown simple graph G (the six subgraphs form a so-called "deck of cards" for unknown graph G).

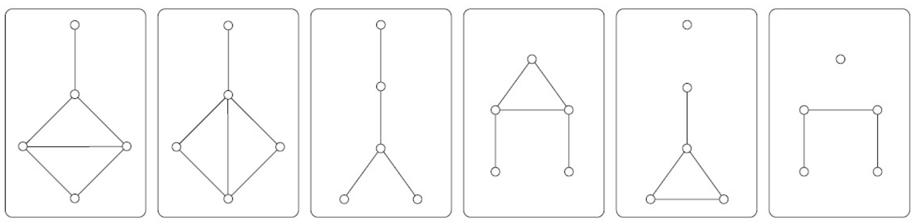
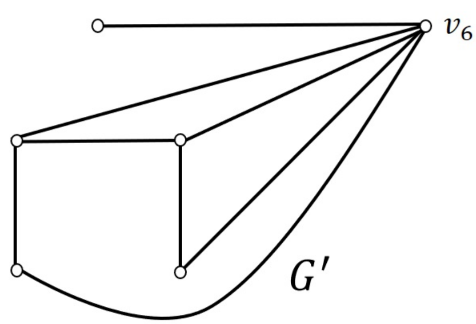
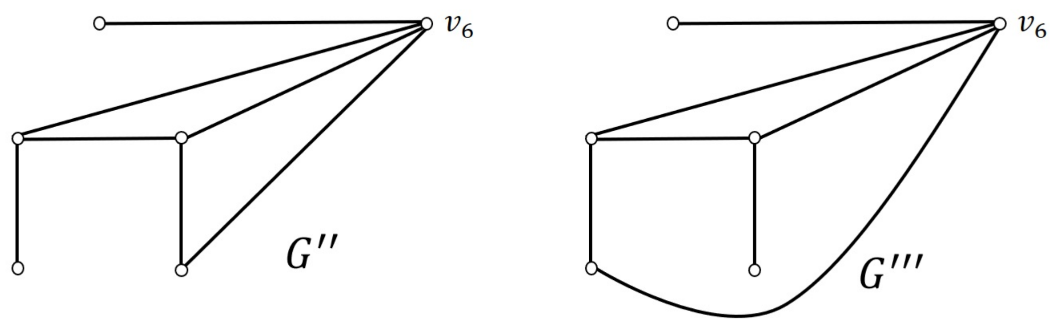


Figure 2.12. The "deck of cards" for a graph on six vertices.

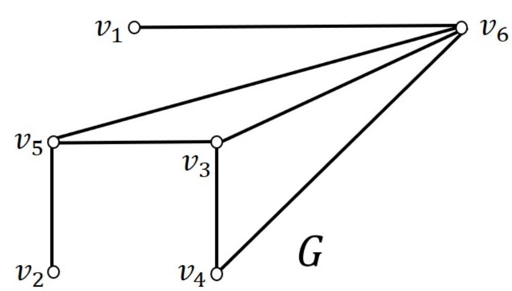
Example. We now reconstruct graph G from the given deck of cards given in Figure 2.12. From "Card #6" (the vertex-deleted graph with the fewest edges), we introduce a new vertex adjacent to all the other vertices (so that G is a subgraph of this new graph):



Graph G' has 8 edges. We try to produce the graph of Card #5. But the graph of Card #5 has four edges so we need to delete a vertex of degree four. But G' doesn't have a degree four vertex; it has a degree 5 vertex and the other vertices are degrees 1, 2, and 3. So we need to eliminate an edge from v_6. Also in the graph of Card #5, there is an isolated vertex so in the desired reconstructed graph this isolated vertex must have been adjacent only to one vertex. This gives us two choices on edge removal:



But G'' and G''' are isomorphic. We can use either version and label as follows:



Then with $G = G''$ we have that $G - v_i$ gives the graph of Card $\#i$. So G is the graph reconstructed from the deck of six cards given in Figure 2.12.

Note. In 1964, Frank Harary published the following concerning graph reconstruction:

Conjecture 2.19. THE RECONSTRUCTION CONJECTURE.

Every simple graph on at least three vertices is reconstructible.

In 1977, B. D. McKay verified the Reconstruction Conjecture for all graphs on ten or less vertices (in the first volume of *The Journal of Graph Theory*!).

Definition. A class of graphs is a *reconstructible class* if every member of the class is reconstructible. A graph parameter is a *reconstructible parameter* if the parameter takes the same value on all reconstructions of G.

Note. In Exercise 2.7.5, it is to be shown that the class of regular graphs is a reconstructible class. In Exercise 2.7.11 it is to be shown that the class of disconnected graphs is a reconstructible class.

Note. To show that a graph parameter on G is reconstructible, we can show that it is a function of the same parameters for the graphs $G - v$ (as v ranges over V). An example of this is given in the next lemma. It concerns the parameter $\binom{G}{F}$ which is the number of copies of F in G (that is, the number of subgraphs of G isomorphic to F). For example, if $F = K_2$ then $\binom{G}{F} = e(G)$ and if $F = g$ then $\binom{G}{F} = 1$.

Lemma 2.20. KELLY'S LEMMA.

For any two graphs F and G such that $v(F) < v(G)$, the parameter $\binom{G}{F}$ is a reconstructible parameter.

Note. It may seem odd to consider the parameter $\binom{G}{F}$ in Kelly's Lemma, but we are about to get some mileage out of it.

Corollary 2.21. For any two graphs F and G, the number of subgraphs of G that are isomorphic to F and includes a given vertex v is a reconstructible parameter.

Corollary 2.22. The size and the degree sequence are reconstructible parameters.

Definition. A graph is *edge-reconstructible* if it can be reconstructed (up to isomorphism) from its edge-deleted subgraphs. An *edge-reconstructible class* and *edge-reconstructible parameter* are similarly defined to the corresponding terms in the vertex-deleted subgraph setting.

Note. The concept of edge reconstructible was introduced by Frank Harary in 1964. As with (vertex) reconstruction, we have the following conjecture.

Conjecture 2.23. The Edge Reconstruction Conjecture.
Every simple graph on at least four edges is edge-reconstructible.

Note. In Exercise 2.7.2, examples of non-edge-reconstructible graphs with two and with three edges are to be given, showing the lower bound in the Edge Reconstruction Conjecture is necessary. A version of Kelly's Lemma (Lemma 2.20) holds in the edge reconstruction, as follows. The proof is to be given in Exercise 2.7.13(a).

Lemma 2.24. Kelly's Lemma: Edge Version.
For any two graphs F and G such that $e(F) < e(G)$, the parameter $\binom{G}{F}$ is edge reconstructible.

Note. In Exercise 2.7.14(b), it is to be shown that the (vertex) Reconstruction Conjecture implies the Edge Reconstruction Conjecture. This makes intuitive sense since a vertex deletion to equivalent to a sequence of edge deletions. As Bondy and Murty put it, "edge-deleted subgraphs are much closer to the original graph than are vertex-deleted subgraphs" (see page 68). As a consequence, there are approaches to establishing edge reconstruction which are not applicable to vertex reconstruction. One of these is Möbius Inversion.

Note. The Inclusion Exclusion formula involves expressing the union of a finite collection of not-necessarily-disjoint sets as a union of disjoint sets. This arises in probability when dealing with the probability of a union of sets. For a statement of the Inclusion Exclusion Formula in the probability setting, see Theorem 1.3.B in my online notes for Mathematical Statistics 1 (MATH 4047/5047) on 1.3. The Probability Set Function; for a proof, see Theorem 1.10.2 in my online notes for Intermediate Probability and Statistics (not a formal ETSU class) on 1.10. The Probability of a Union of Events. In our current setting, we consider only finite sets and so we can state the formula as follows.

Theorem 2.7.A. The Inclusion Exclusion Formula.

Let $A_1, A_2, \ldots, A_n$ be a collection of finite sets and let the index set be $T = \{1, 2, \ldots, n\}$. Then

$$|\cup_{i \in T} A_i| = \sum_{\varnothing \subsetneq X \subseteq T} (-1)^{|X|-1} |\cap_{i \in X} A_i| \,.$$

Note. In the case $T = \{1, 2\}$, we have the familiar relationship $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. If $T = \{1, 2, 3\}$ we have (see Theorem 1.10.1 in 1.10. The Probability of a Union of Events):

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Notice that Bondy and Murty do not explicitly state that the sets must be finite, but this is the case since subtraction of infinite cardinal numbers is not (necessarily) defined. Also, Bondy and Murty do not state that the index set T is finite (it could be countably infinite, but then the convergence of the series that results becomes a concern). Notice that they do impose $|T| < \infty$ in the next theorem.

Theorem 2.25. The Möbius Inversion Formula.

Let $f : 2^T \to \mathbb{R}$ (here, 2^T represents the power set of T, $2^T = \mathcal{P}(T)$) be a real-valued function defined on the subsets of a finite set T. Define the function $g : 2^T \to \mathbb{R}$ by $g(S) = \sum_{S \subseteq X \subseteq T} f(X)$. Then for all $S \subseteq T$,

$$f(S) = \sum_{S \subseteq X \subseteq T} (-1)^{|X|-|S|} g(X).$$

Note. In the Möbius Inversion Formula, function g is a linear transformation on the vector space of real-valued functions defined on 2^T and f is its inverse. This is the reason for the name of Theorem 2.25. We'll use the Möbius Inversion Formula to give sufficient conditions for a graph to be edge reconstructible.

Note. In dealing with edge reconstructibility, we count the mappings between two simple graphs G and H (on the same vertex set V) based on the intersection of the image of G, $\sigma(G)$, and H. The mappings σ are just permutations of V where we take $\sigma(G) = (V, \sigma(E))$ for $\sigma(E) = \{\sigma(u)\sigma(v) \mid uv \in E\}$. For each spanning subgraph F of G, we want to count the permutations of G which map the edges of F onto edges of H and which map the remaining edges of G onto edges of $\overline{H}$ (the complement of graph H; see Exercise 1.1.17). We denote the number of such permutations as $|G \to H|_F$ so that $|G \to H|_F = |\{\sigma \in S_n \mid \sigma(G) \cap H = \sigma(F)\}|$ where S_n is the symmetric group on n symbols (a group of order $|S_n| = n!$). Notice that for $F = G$, $|G \to H|_F = |G \to H|_G$ is just the number of embeddings of G in H, which we abbreviate $|G \to H|$. If F is empty then $|G \to H|_F = |G \to H|_\varnothing$ is the number of embeddings of G in $\overline{H}$, $|G \to \overline{H}|$. In Figure 2.13, we have $G = K_1 + K_{1,2}$ and $H = 2K_2$ and for all subgraphs F of G we give $|G \to G|_F$ and $|G \to H|_F$.

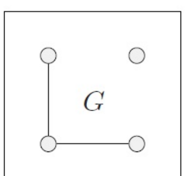
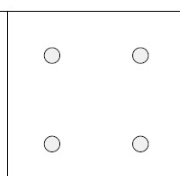
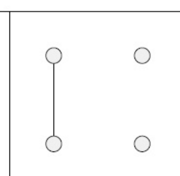
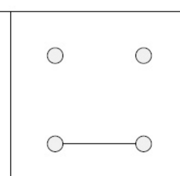
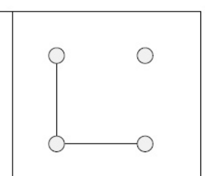
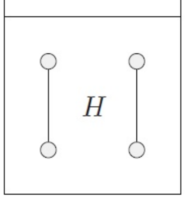
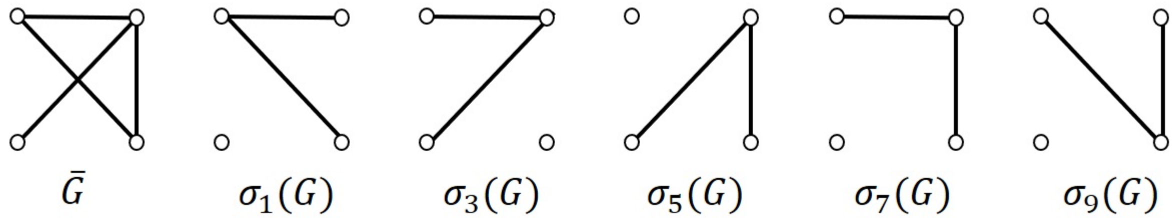
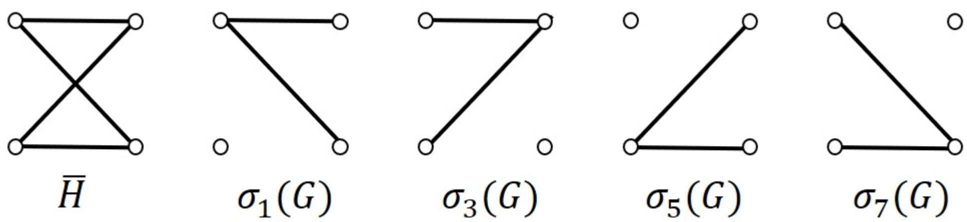
	F				
	$\vert G \to G\vert_F$	10	6	6	2
	$\vert G \to H\vert_F$	8	8	8	0

Figure 2.13

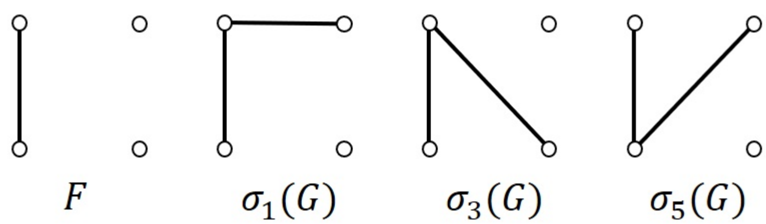
Notice that when $F = \varnothing$ then $|G \to G|_F$ is then number of embeddings of G in $\overline{G}$,



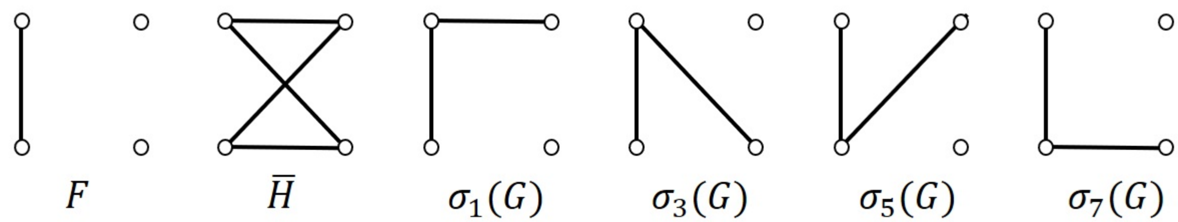
and we can interchange the degree on vertices in the images of G to produce another five appropriate permutations, so $|G \to G|_F = 10$. When $F = \varnothing$ then $|G \to H_F$ is the number of embeddings of G in $\overline{H}$,



and we can interchange the degree on vertices in the images of G to produce another four appropriate permutations, so $|G \to H|_F = 8$. When F is a single edge of G we have (for example),



and we can interchange the degree one vertices as above to produce another three appropriate permutations, so $|G \to G|_F = 6$. When F is a single edge of G we have (for example),



and we can interchange the degree one vertices as above to produce another four appropriate permutations, so $|G \to H|_F = 8$. If $F = G$ then $|G \to G|_F$ is the number of embeddings of G in G (i.e., the of automorphisms of G) and $|G \to G|_G = 2$. If $F = G$ then $|G \to H|_F$ is the number of embeddings of G in H and so $|G \to H|_F = 0$.

Note. For a given F a subgraph of G, $|F \to H|$ is the number of embeddings of F in H. Since the embeddings are elements of S_n, then an embedding of F in H must map the edges of F to the edges of H, map some of the edges of G (those not in F) to H, and map some of the other edges of G (those not in F and those not mapped to H) to $\overline{H}$. For each such embedding, there is a graph X with $F \subseteq K \subseteq G$ where the edges of X are the edges of F combined with the edges of G which are mapped to H under the embedding. For this given X, there are $|G \to H|_X$ mappings with this property. So the total number of embeddings of F in H is

$$|F \to H| = \sum_{F \subseteq X \subseteq G} |G \to H|_X. \tag{2.6}$$

Note. Since $\binom{H}{F}$ denotes the number of copies of F in H and $\mathrm{aut}(F)$ is the number of automorphisms of F, then

$$|F \to H| = \mathrm{aut}(F)\binom{H}{F} \tag{2.7}$$

since for each of the $\binom{H}{F}$ copies of F in H, F can be mapped to the copy of F in H a total of $\mathrm{aut}(F)$ distinct ways.

Lemma 2.26. NASH-WILLIAMS' LEMMA.
Let G be a graph, F a spanning subgraph of G, and H an edge reconstruction of G that is not isomorphic to G. Then

$$|G \to G|_F - |G \to H|_F = (-1)^{e(G)-e(F)}\mathrm{aut}(G).$$

Note. Notice that Nash-William's lemma concerns *an* edge reconstruction H of G that is *not* isomorphic to G. If this condition holds, then G is not reconstructible. So we use the Nash-Williams' Lemma to give some sufficient conditions for the reconstructilbility of a graph G by violating the lemma (and concluding that H is isomorphic to G).

Theorem 2.27. A graph G is edge reconstructible if there exists a spanning subgraph F of G such that either of the following two conditions holds:

(i) $|G \to H|_F$ takes the same value for all edge reconstructions H of G.

(ii) $|F \to G| < 2^{e(G)-e(F)-1}\mathrm{aut}(G)$.

Note. The proof of the following sufficient condition for edge reconstructibility is to be given in Exercise 2.7.8; the proof is based on taking $F = \varnothing$ in Theorem 2.27.

Corollary 2.28. A graph G of size m and order n is edge reconstructible if either $m > \frac{1}{2}\binom{n}{2}$ or $2^{m-1} > n!$.

Revised: 3/29/2020