Chapter 3. Connected Graphs

Section 3.1. Walks and Connection

Note. We define walks in a graph; paths are closely related to walks. We use paths to define a measure of distance on a graph, and state and prove "The Friendship Theorem."

Definition. A *walk* in a graph G is a sequence $W = v_0e_1v_1e_2v_2 \cdots v_{\ell-1}e_\ell v_\ell$ whose terms are alternatively vertices and edges of G (not necessarily distinct), such that v_{i-1} and v_i are the ends of e_i for $1 \le i \le \ell$. If $v_0 = x$ and $v_\ell = y$ then W *connects* x to y and W is called an *xy-walk*. The vertices x and y are the *ends* of the walk with x as the *initial vertex* and y as the *terminal vertex*, and vertices $v_1, v_2, \ldots, v_{\ell-1}$ are *internal vertices*. The number of edges in a walk W (ℓ in our notation here) is the *length* of W. An *x-walk* is a walk with initial vertex x. If u and v are vertices of a walk W, where u precedes v in W, the subsequence of W starting at u and ending at v is the *segment* of W from u to v, denoted uWv. A walk is *closed* if its initial and terminal vertices are the same. A walk is a *trail* if all its edges are distinct.

Note. In a simple graph a walk $v_0e_1v_1 \cdots v_{\ell-1}e_\ell v_\ell$ is determined by its vertices and in this case we may denote the walk as $v_0v_1 \cdots v_\ell$.

Note. It is straightforward to show that connectedness of pairs of vertices (by a walk) is an equivalence relation (see page 80). In Exercise 3.1.3 it is to be shown that the equivalence classes of this equivalence relation are the connected components of a graph. In Exercise 3.1.1 it is to be shown that if a graph has an xy-walk then it has an xy-path. This observation allows us to define distance in a graph.

Definition. If vertices x and y are in the same component of graph G then the *distance* between x and y is the length of the shortest path between x and y, denoted $d_G(x, y)$. If x and y are in different components of G when we define $d_G(x, y) = \infty$. The *diameter* of a graph G is the greatest distance between two vertices of G:

$$\mathrm{diam}(G) = \max\{d_G(x, y) \mid x, y \in V(G)\}.$$

Note. It follows from Exercise 3.1.5 (the Triangle Inequality for d_G) that d_G is a metric on the components of a graph.

Definition. Let X and Y be subsets of $V(G)$. An (X, Y)-*path* in G is a path which starts at a vertex of X, ends at a vertex of Y, and whose internal vertices belong to neither X nor Y.

Note 3.1.A. In Exercise 3.1.4 it is to be shown that graph G is connected if and only if for any nonempty $X, Y \subseteq V(G)$ there is an (X, Y)-path in G. Notice that in a connected graph with vertices x and y, with $X = \{x\}$ and $Y = \{y\}$ we see by Exercise 3.1.4 that the graph must contain a path joining vertices x and y. Conversely if for any vertices x and y in a graph there is an xy-path in the graph, then for any nonempty sets $X, Y \subseteq V(G)$ there is an (X, Y)-path in G so that by Exercise 3.1.4 G is connected. So we could have taken the definition of "connected graph" as a graph such that for any two vertices of the graph, there is a path joining those vertices in the graph. This is a standard way to define "connected graph" (as opposed to our approach in Section 1.1 where we defined a graph as connected if for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and one end in Y). In fact, in Bondy and Murty's *Graph Theory with Applications* (North Holland Press, 1976), connectivity is defined in terms of paths joining vertices (see page 13; this book can viewed online at web.archive).

Definition. A *friendship graph* is a simple graph in which any two vertices have exactly one common neighbor.

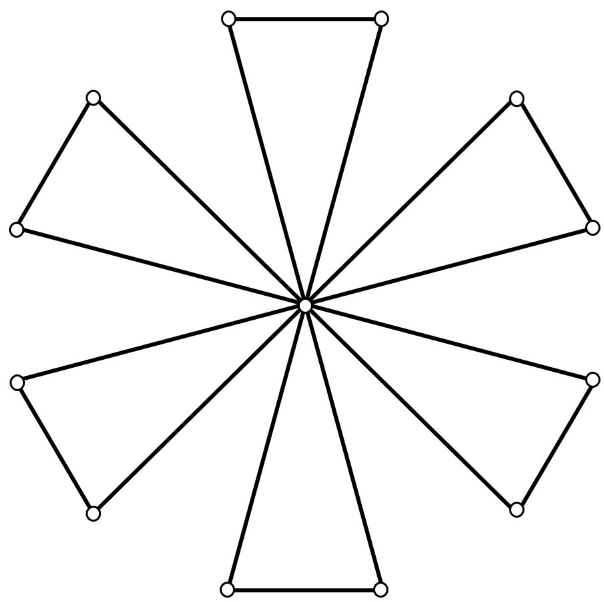
Note. We will use distances in graphs and eigenvalues of graphs to prove "The Friendship Theorem." First, we need a preliminary lemma.

Lemma 3.1.A. Let $\mathbf{J}$ be the $n \times n$ matrix with all entries 1. Then the eigenvalues of $\mathbf{J}$ are 0 (with algebraic multiplicity $n-1$) and n (with algebraic multiplicity 1).

Theorem 3.1. THE FRIENDSHIP THEOREM.

Let G be a simple graph in which any two vertices (people) have exactly one common neighbor (friend). Then G has a vertex of degree $n-1$ (everyone's friend).

Note. This result was first proved in Paul Erdős, Alfréd Rényi, and Vera T. Sós, "On a Problem of Graph Theory," *Studia Scientiarum Mathematicarum Hungarica*, **1** (1966), 215–235. A copy is available online on the Collected papers of Paul Erdős webpage of the Alfréd Rényi Institute of Mathematics (Hungarian Academy of Sciences); see Theorem 6 on page 234 (accessed 11/11/2022). They actually give the structure of a Friendship Graph as "k triangles which have one common vertex." Such graphs are sometimes called "Dutch windmill graphs," denoted D_3^k. The Dutch windmill D_3^6 (with 6 triangles) is given below.



Note. A proof of The Friendship Theorem (Theorem 3.1), equivalent to the Bondy and Murty proof, is also given in Martin Aigner and Günter Ziegler's *Proofs from THE BOOK*, 6th edition, Springer (2018). See their Chapter 44, "Of Friends and Politicians."