Section 3.3. Euler Tours

Note. The earliest known paper on graph theory is by Leonard Euler: "Solutio problematis ad geometriam situs pertinentis" (Solution to the geometry of position), *Comment. Academiae Sci. I. Petropolitanae* **8** (1736), 128–140.

Leonhard Euler (April 15, 1707–September 18, 1783)

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The paper addressed the Kónigsberg Bridge Problem. The Pregel River runs through Königsberg, formerly in Germany but now in Russia and called Kaliningrad (see the brief WolframMathWorld article on the problem; accessed 11/14/2022). A diagram of the river and the associated graph appears in Figure 3.4.

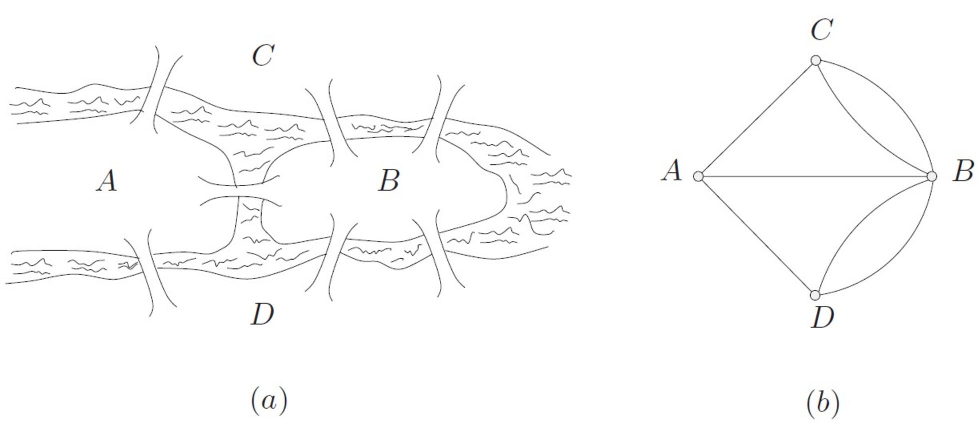


Figure 3.4

The problem is to find a trajectory to walk across the bridges such that each bridge is crossed just once. By representing the land areas as vertices and the bridges as edges of a graph, as given in Figure 3.4(b), the question can be translated into graph theoretic terms concerning the existence of a walk through the graph with certain properties. This inspires the following definitions.

Definition. A *tour* of a connected graph G is a closed walk that includes each edge of G at least once. An *Euler tour* is a tour that includes each edge exactly once. A graph is *Eulerian* if it admits an Euler tour. An *Euler trail* is a trail (i.e., edges are distinct, ends may not be the same) that includes each edge of G.

Lemma 3.3.A. If G is an Eulerian graph then G is even.

Note. We'll see below that the converse of Lemma 3.3.A also holds and that a connected graph is Eulerian if and only if it is even. The fact that G is even is sufficient for G to be Eulerian is established by Fleury's Algorithm. This was presented by M. Fleury in "Deux problèmes de géométric de situation," *Journal de Mathématiques Elémentaires* (1883), 257–261. This is available (in French) on Google Books (accessed 11/14/2022). Fleury's Algorithm constructs an Euler tour by tracing out a trail under the condition that at each stage a cut edge of the untraced subgraph is taken only if there is no other edge choice. Bondy and Murty present the algorithm in a format that reminds *me* of the style of Fortran (with a "`do-while`" loop), as follows.

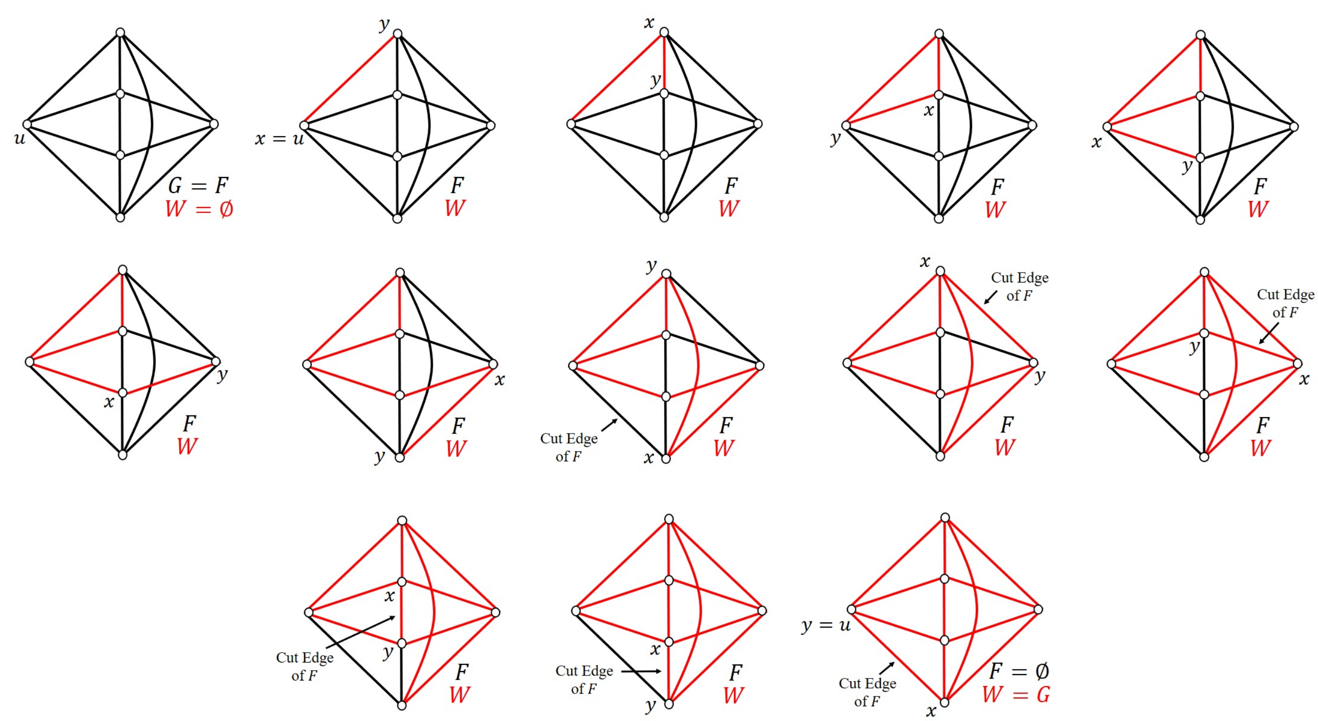
Algorithm 3.3. FLEURY'S ALGORITHM.

INPUT: a connected even graph G and specified vertex u of G

OUTPUT: an Euler tour W of G starting and ending at u

1. *set* $W := u,\ x := u,\ F := G$
2. **while** $\partial_F(x) \neq \varnothing$ **do**
3. *choose an edge* $e := xy \in \partial_F(x)$, *where* e *is not a cut edge of* F *unless there is no alternative*
4. *replace* uWx *by* $uWxey$, y *by* x, *and* F *by* $F \setminus e$
5. **end while**
6. *return* W.

Note. We illustrate Fleury's Algorithm as follows:



Note. The next theorem shows that Fleury's Algorithm actually works. The presented proof may appear novel to you, unless you have dealt with arguments involving algorithms before.

Theorem 3.4. If G is a connected even graph, then the walk W returned by Fleury's Algorithm is an Euler tour of G.

Note. Lemma 3.3.A and Theorem 3.4 combine to classify Eulerian graphs as follows.

Theorem 3.5. A connected graph G is Eulerian if and only if G is even.

Note. Theorem 3.5 now shows that there is no solution to the Kónigsberg Bridge Problem, since the graph associated with the problem has every vertex of odd degree (see Figure 3.4(b)).

Note. Predating Fluery's algorithm is an argument given by Carl Hierholzer (October 2, 1840–September 13, 1871). He discussed it with his colleagues, but died unexpectedly shortly afterwards. One of them arranged for the result to be published and it appeared in: Carl Hierholzer and Chr. Wiener, "Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren [On the Possibility of Traversing a Line-System without Repetition or Discontinuity],” *Mathematische Annalen*, **6**, 30-32 (1873). This brief work is reprinted in English in N. L. Biggs, E. K. Loyd, and R. J. Wilson’s *Graph Theory: 1736-1936* (Oxford: Oxford University Press, 1976); see pages 11 and 12. The proof of this result is today called “Hierholzer’s algorithm.” For more details and a proof, see my online notes for Introduction to Graph Theory (MATH 4347/5347) on Section 3.1. Eulerian Circuits.

Note. In Exercise 3.3.3 it is to be shown how Fleury’s Algorithm can be modified to find an Euler trail between two vertices x and y of a connected graph G where vertices x and y are of odd degree, but all other vertices of G are of even degree. In Exercise 3.3.4(b), the following is to be shown.

Theorem 3.3.A. Let G be a connected graph. There are vertices $x, y \in V$ where $d(x)$ and $d(y)$ are odd and $d(v)$ is even for all $v \in V \setminus \{x, y\}$ if and only if G has a Euler trail connecting x and y.

Note. Bondy and Murty mention a two volume work by Herbert Fleischner: *Eulerian Graphs and Related Topics, Part 1, Volume 1*, *Annals of Discrete Mathematics*, Vol. 45, Amsterdam: North-Holland (1990), and *Eulerian Graphs and Related Topics, Part 1, Volume 2*, *Annals of Discrete Mathematics*, Vol. 50, Amsterdam: North-Holland (1991).

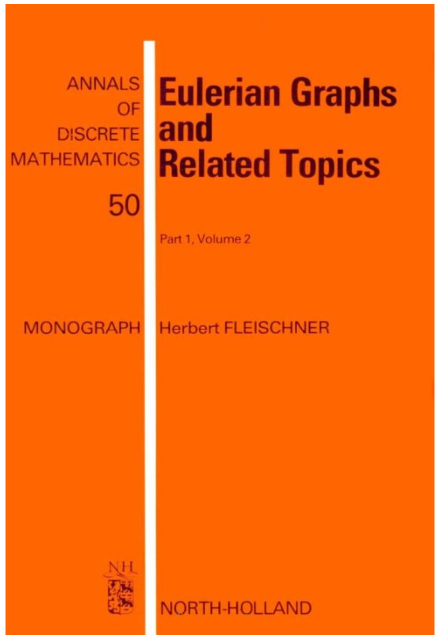


Image from Amazon (accessed 3/13/2020)

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