Section 3.4. Connection in Digraphs

Note. We extend the ideas of walks, paths, trails, tours, Euler trails, Euler tours, and the property of Eulerian to digraphs.

Definition. A *directed walk* in a digraph D is an alternating sequence of vertices and arcs,

$$W = (v_0, a_1, v_1, a_2, v_2, \ldots, v_{\ell-1}, a_\ell, v_\ell)$$

such that vertices v_{i-1} and v_i are the tail and head of arc a_i, respectively, for $1 \leq i \leq \ell$. If x and y are the initial and terminal vertices of W then W is called a *directed* (x, y)*-walk*. *Directed trails*, *directed tours*, *directed paths*, and *directed cycles* in digraphs are defined in a way analogous to the graph setting. Vertex y is *reachable* from vertex x in a digraph if there is a directed (x, y)-path in the digraph.

Theorem 3.6. Let x and y be two vertices of a digraph D. Then y is reachable from x in D if and only if the outcut $\partial^+(X) \neq \varnothing$ for every subset X of V which contains x but not y.

Definition. In a digraph D, two vertices of x and y are *strongly connected* if both x is reachable from y and y is reachable from x.

Note. In Exercise 3.4.1 it is to be shown that strong connection is an equivalence relation on the vertex set of a digraph.

Definition. The subgraphs of a digraph D which are induced subgraphs of equivalence classes of the digraph under the equivalence relation of strong connection are called the *strong components* of the digraph.

Note. Figure 3.7 gives a digraph and its strong components. Notice that the components include all vertices of the digraph (because strong connection is an equivalence relation on the vertex set of the digraph), but do not include all arcs of the digraph.

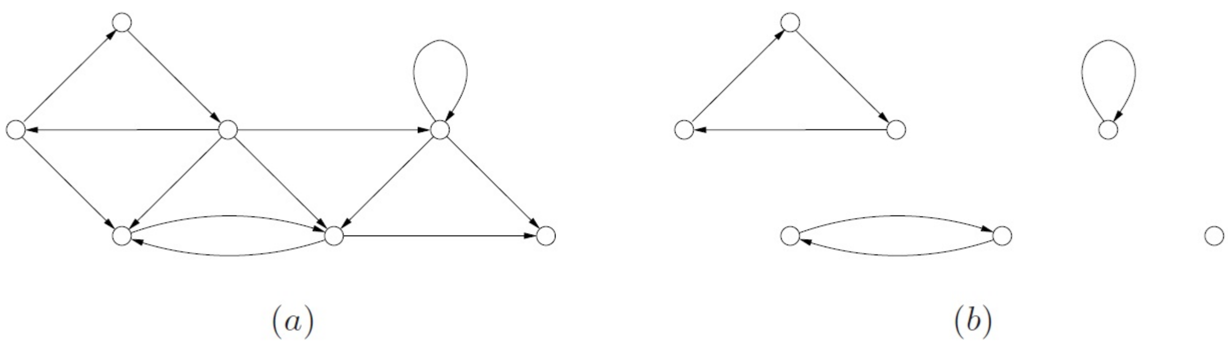

Figure 3.7. (a) A digraph and (b) its strong components.

Definition. A *directed Euler trail* is a directed trail which traverses each arc of the digraph exactly once. A *directed Euler tour* is a directed term which traverses each arc of the digraph exactly once. A digraph is an *Eulerian digraph* if it admits a directed Euler tour.

Note. The proof of the following result (which is a directed version of Theorem 3.5) is to be proved in Exercise 3.4.8.

Theorem 3.7. A connected digraph is Eulerian if and only if it is even.

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