Section 3.5. Cycle Double Covers

Note. We consider cycle covers of a graph G in which every edge of G appears twice in the cover. We state the Cycle Double Cover Conjecture and look at cycle double covers of planar graphs and graphs which can be embedded in surfaces. We also state two other related conjectures.

Note. Recall from Section 2.4, "Decompositions and Coverings," that a decomposition of a graph G is a partitioning of the edge set of G into the edge sets of edge-disjoint subgraphs of G, and a covering is a collection of (not necessarily edge-disjoint) subgraphs of G the union of whose edge sets is the edge set of G. By Veblen's Theorem (Theorem 2.7), a cycle decomposition of G exists if and only if G is even. So if G has a vertex of odd degree then a cycle cover of G must include some edge of G more than once. "One is led to ask whether every graph without cut edges admits a cycle covering in which no edge is covered more than twice" (Bondy and Murty, page 93). By Proposition 3.2, an edge e of a graph G is a cut edge if and only if e belongs to no cycle of G, so the condition of considering cycle coverings of graphs without cut edges is necessary (since in coverings, we only consider subgraphs of G; this restriction could be lifted and we could consider "unrestricted coverings," as in R. Gardner and W. Surber, Restricted and Unrestricted Hexagon Coverings of the Complete Bipartite Graph, *Congressus Numerantium*, **217** (2013), 107-128).

Note. Figure 3.8 gives a 4-cycle covering of the cube in which each edge is covered exactly twice.

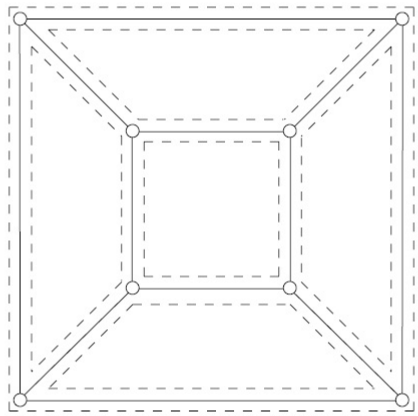

Figure 3.8. A 4-cycle double covering of the cube by its "facial cycles."

Definition. A cycle covering of a graph G in which each edge of G is covered twice is a *cycle double cover* of G.

Note. A cycle double cover of the cube is given in Figure 3.8. Cycle coverings and cycle double coverings are closely related, as follows.

Proposition 3.8. If a graph has a cycle covering in which each edge is covered at most twice, then it has a cycle double cover.

Note. In 1973, George Szerkeres conjectured (and Paul Seymour repeated the conjecture in 1979) that the obvious necessary condition for the existence of a cycle double cover is in fact sufficient.

Conjecture 3.9. The Cycle Double Cover Conjecture.

Every graph without cut edges has a cycle double cover.

Note. In the case that graph G is planar, then a cycle double cover can be found by taking the "facial cycles" that bound each region determined by a planar drawing of the graph. This is illustrated for the cube in Figure 3.8 above and is addressed more rigorously in Chapter 10. In fact, this technique could be used for graphs that can be embedded on a surface (not necessarily a plane). In Figure 3.9, embeddings of K_7 and the Petersen graph into a torus are given (a graph that can be embedded on a torus is a *toroidal graph*). Interpret the figure by conceptually joining the left and right sides of the square (making cylinders) and then conceptually joining the tops and bottoms of the cylinders (making tori). There are 14 "faces" of the torus determined by K_7 and so 14 3-cycles (the facial cycles) yield a cycle double cover of K_7. There are 5 faces of the torus determined by the Petersen graph (regions D and E are cut in Figure 3.9(b). The facial cycles of the Petersen graph consist of 3 5-cycles (for faces A, B, and C), a 6-cycle (for face D), and a 9-cycle (for face E).

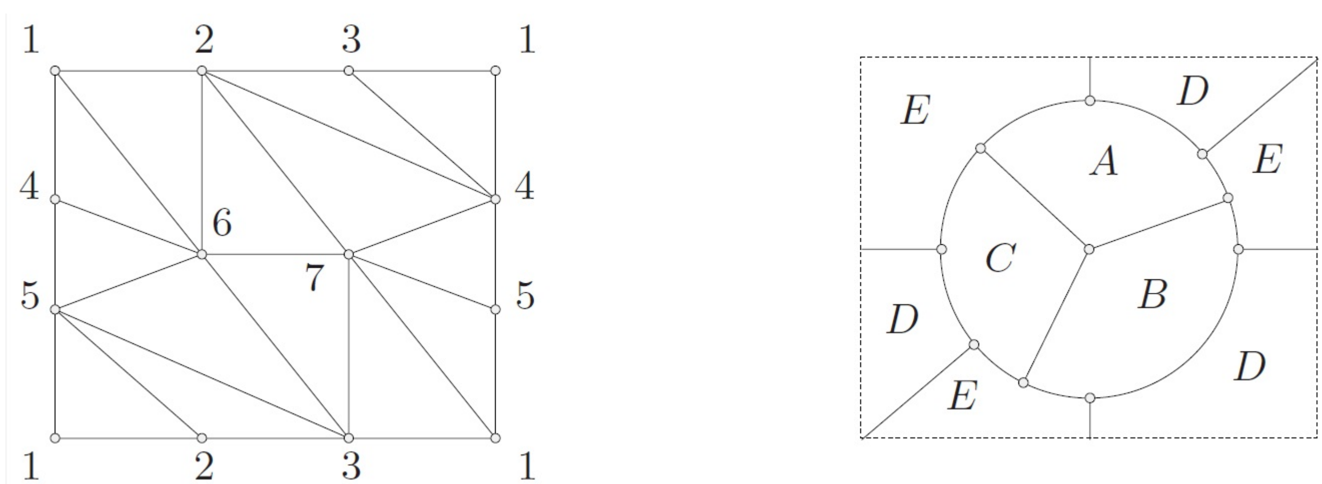

Figure 3.9

Note. The previous note inspires an explanation of embeddings of graphs on surfaces and the following conjecture. It deals with "nonseparable graphs," which are formally defined in Section 5.2. Quoting from the text (see page 94): "Roughly speaking, these are the connected graphs which cannot be obtained by piecing together two smaller connected graphs at a single vertex."

Conjecture 3.10. THE CIRCULAR EMBEDDINGS CONJECTURE.
Every loopless nonseparable graph can be embedded in some surface in such a way that each face in the embedding is bound by a cycle.

Note. Under the assumption that the Cycle Double Cover Conjecture holds, one could address how few cycles are necessary for a cycle double cover. Let $\mathcal{C}$ be a cycle double cover of a graph G. Since each edge of G is covered twice, we have $\sum_{C \in \mathcal{C}} e(C) = 2|E| = 2m$. Now $e(C) \leq n$ for all $X \in \mathcal{C}$, so we must have $n|\mathcal{C}| \geq 2m$ or $|\mathcal{C}| \geq 2m/n$ and $2m/n$ is the average degree of the vertices of G (i.e., $2m$ edge ends divided by the number of vertices).

Definition. A cycle double cover of a graph G consisting of at most $2m/n$ cycles is a *small cycle-double-cover.*

Note. Notice that in a small cycle-double-cover, it is the *number* of cycles that is small, not the *length* of the cycles. Two other open conjectures concerning cycle double covers are the following.

Conjecture 3.11. [Bondy 1990] The Small Cycle Double Cover Conjecture.

Every simple graph without cut edges has a small cycle double cover.

Conjecture 3.12. [Jaeger 1988] The Oriented Cycle Double Cover Conjecture.

Let G be a graph without cut edges. Then the associated digraph $D(G)$ of G admits a decomposition into directed cycles of length at least three.

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