Chapter 4. Trees

Section 4.1. Forests and Trees

Note. In this section we define tree, forest, and branching (or "arborescence"), give some elementary properties, and explore branchings as subgraphs of tournaments.

Definition. A connected acyclic graph is a *tree*. An acyclic (not necessarily connected) graph is a *forest*.

Note. Figure 4.1 gives all trees ("up to isomorphism," that is) on six vertices. This figure can also be interpreted as a forest with six connected components.

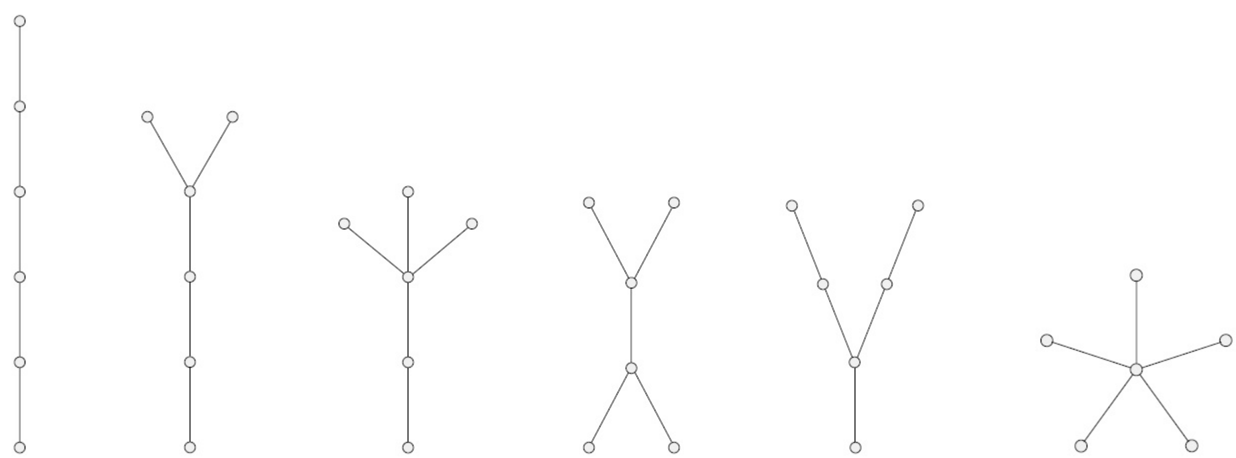


Figure 4.1

Proposition 4.1. In a tree, any two vertices are connected by exactly one path.

Note. For vertices x and y in tree T, we denote the unique path connecting x and y of Proposition 4.1 as xTy or yTx. This is the notation used by Reinhard Diestel in *Graph Theory*, Graduate Texts in Mathematics #173 5th edition, Springer (2017); an earlier version of this book is available from the ETSU Sherrod Library (QA166.D51413 1997) and the 2000 version is available through the library for online access. Diestel's book is a competitor with our text book for use in a class at this level.

Definition. A vertex in a tree of degree one is a *leaf*.

Note. The contrapositive of Theorem 2.1 states that a nontrivial tree has at least one leaf. In fact, Exercise 2.1.2 can be reworded as follows.

Proposition 4.2. Every nontrivial tree has at least two leaves.

Theorem 4.3. If T is a tree, then $e(T) = v(T) - 1$.

Definition. A *rooted tree* is a tree T with a specified vertex x called the *root* of T; such a rooted tree is denoted $T(x)$ and called an *x-tree*. An orientation of a rooted tree in which every vertex has indegree 1, except for the root which has indegree 0, is a *branching*. A branching with root x is called an *x-branching*.

Note. Figure 4.2 gives an example of a branching where the root vertex is shaded grey. Notice that if we reverse the direction of any arc in Figure 4.2 then we no longer have a branching. The term "branching" is not universal. A common replacement is the term "arborescence" (French for "tree"). W. T. Tutte in *Graph Theory*, Cambridge University Press (2001) uses the term "arborescence diverging from" where we use the term branching and he uses the term "arborescence converging to" to represent the orientation of a rooted tree where each arc is reversed from the direction given in a branching (so each vertex is outdegree 1, except for the root which is outdegree 0). See his pages 126 and 127.

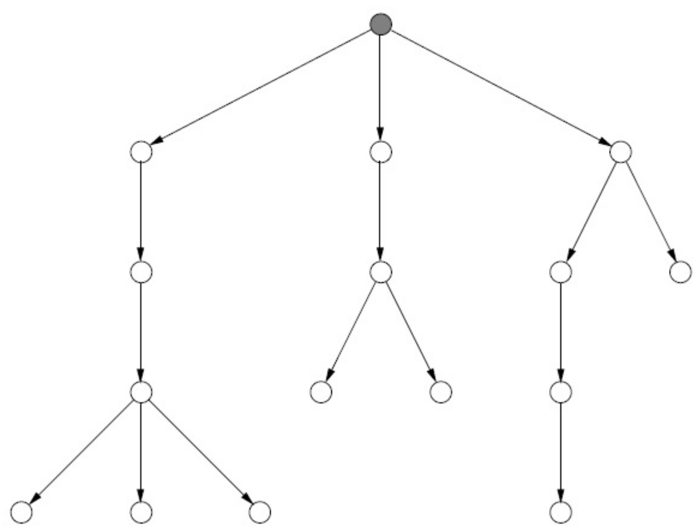


Figure 4.2

Note. Notice that each vertex of a branching is reachable from the root by a (unique) directed path. The next result expresses the set of vertices in a digraph which are reachable from a given vertex x in terms of an x-branching subdigraph. The proof is to be given in Exercise 4.1.6.

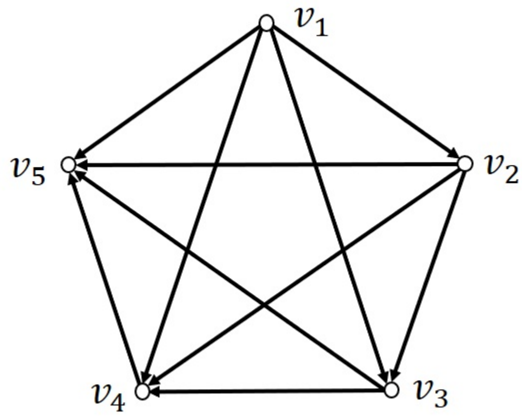
Theorem 4.4. Let x be a vertex of a digraph D, and let X be the set of vertices of D which are *reachable* from x. Then there is a branching in D with vertex set X.

Note. In Exercise 4.1.9 it is to be shown that every simple graph G with $\delta(G) = k$ contains a copy of each rooted tree on $k + 1$ vertices, rooted at any given vertex of G. Informally, this means that graphs contain lots of different trees. A formula for the number of nonisomorphic trees on n vertices is not know, but if the vertices are labeled then there are n^{n-2} nonisomorphic trees on the n labeled vertices; this is Cayley's Formula which we will see in the next section (see Theorem 4.8). We addressed labled graphs in Section 1.2 (see Figure 1.10, for example).

Note. The remainder of this section deals with ordering vertices, median order, and branchings in a tournament. If pressed for time, this material may be skipped without impacting the rest of the course.

Definition. A *median order* or a digraph $D = (V, A)$ is a linear order $v_1, v_2, \ldots, v_n$ of its vertex set V such that $|\{(v_i, v_j) \mid 1 \leq i < j \leq n\}|$ (the number of arcs directed left to right in the linear order) is as large as possible.

Note. We can think of the linear ordering as the ranking of teams, all playing each other in an athletic tournament. The median order represents the ranking that minimized the number of "upsets," where a lower ranked team defeats a higher ranked team. The median order of the rank in the figure below is v_1, v_2, v_3, v_4, v_5 since $|\{(v_i, v_j) \mid 1 \le i < j \le 5\}| = 10$.



Note. The following properties of median orders of tournaments is to be shown in Exercise 4.1.10.

Theorem 4.1.A. Let T be a tournament and $v_1, v_2, \ldots, v_n$ a median order of T. Then for any two indices i and j with $1 \le i < j \le n$ we have:

(M1) the interval $v_i, v_{i+1}, \ldots, v_j$ is a median order of the induced subtournament $T[\{v_i, v_{i+1}, \ldots, v_j\}]$, and

(M2) vertex v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \ldots, v_j$ and vertex v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \ldots, v_{j-1}$.

Note. Theorem 4.1.A(M1) can be used to give an alternative proof of Rédei's Theorem, as follows.

Corollary 4.1.B. Rédei's Theorem.

Every tournament has a directed Hamilton path.

Note. The next result addresses the existence of branchings in a tournament.

Theorem 4.5. Any tournament on $2k$ vertices contains a copy of each branching on $k+1$ vertices.

Note. Rooted trees and branchings will arise again in Chapter 6 when discussing the efficiency of algorithms.

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