# Section 4.2. Spanning Trees

**Note.** In this section we use spanning trees to classify connected graphs and to study bipartite graphs. We state Cayley's Formula which gives the number of spanning trees of $K_n$.

**Definition.** A *subtree* of a graph is a subgraph which is a tree. A subtree of a graph which is a spanning subgraph is a *spanning tree*.

**Proposition 4.6.** A graph $G$ is connected if and only if $G$ has a spanning tree.

**Note 4.2.A.** A tree $T$ is a bipartite graph; we can pick any $v \in V(T)$ and then define $X = \{x \in V(T) \mid d_T(x,v) \text{ is even}\}$ and $Y = \{y \in V(T) \mid d_T(y,v) \text{ is odd}\}$, so that $(X,Y)$ is a bipartition of $T$ and $T$ is bipartite. More generally, we have the following.

**Theorem 4.7.** A graph is bipartite if and only if it contains no odd cycle.

**Note 4.2.B.** Recall from Section 1.2 that a labeled simple graph is a simple graph in which the vertices are labeled. Figure 1.10 of Section 1.2 gives the 8 labeled graphs on 3 vertices (notice that they fall into 4 categories by graph isomorphism). We commented in the previous section that there are $n^{n-2}$ trees on $n$ labeled vertices (this is "Cayley's Formula"). When we count the number of subgraphs of a given graph (as we will do soon for the subgraph as a spanning tree), we want to distinguish between isomorphic subgraphs and we do so using the vertex labels. For example, $K_n$ has $\binom{n}{2}$ subgraphs which are isomorphic to $K_2$ BUT each of these is isomorphic to all the others. So the number of labeled trees on $n$ vertices corresponds to the number of spanning trees in $K_n$. There are only 6 nonisomorphic spanning trees of $K_6$ (the 6 trees in Figure 4.1), but there are $6^{(6-2)} = 6^4 = 1296$ spanning trees of $K_6$ (so these 1296 trees fall into 6 isomorphic categories).

**Note.** While considering the number of hydrocarbons of a certain type (those without "cycles"), Arthur Cayley (1821–1895) represented atoms as vertices and chemical bonds as edges (see Exercise 4.1.3). This leads him in 1889 to count the number of labeled trees on $n$ vertices: $n^{n-2}$. The proof of Cayley's Formula we give below is based on J. Pitman "Coalescent Random Forests," *Journal of Combinatorial Theory, Series A* **85**, (1999) 165-193. It uses "branching forests," that is a digraph each of whose components is a branching. An alternative proof is to be given in Exercise 4.2.11 which uses Prüfer codes (this is the proof given in Bondy and Murty's *Graph Theory with Applications*, NY: North-Holland (1976); see my online notes for Introduction to Graph Theory [MATH 4347/5347] on Section 5.2. Cayley's Spanning Tree Formula where this proof is given).

**Theorem 4.8.** CAYLEY'S FORMULA.

The number of labeled trees on $n$ vertices is $n^{n-2}$.

**Note.** We denote the number of spanning trees of graph $G$ as $t(G)$. So, in light of Note 4.2.B and Cayley's Formula, we have $t(K_n) = n^{n-2}$. The following result clearly relates $t(G)$ to $t(G \setminus e)$ ($G \setminus e$ is $G$ with edge $e$ deleted) and $t(G/e)$ ($G/e$ is $G$ with edge $e$ contracted). The proof is to be given in Exercise 4.2.1.

**Theorem 4.9.** Let $G$ be a graph and $e$ a link of $G$. Then $t(G) = t(G \setminus e) + t(G/e)$.

*Revised: 12/8/2022*