Section 4.3. Fundamental Cycles and Bonds

Note. In this section we give "intimate" relationships between the spanning trees of a connected graph, its even subgraphs, and its edge cuts.

Definition. Let G be a connected graph and let T be a spanning tree of G. The complement $E \setminus T$ (where $E = E(G)$) of spanning tree T is called a *cotree*, denoted $\overline{T} = E \setminus T$.

Note 4.3.A. The "co" in cotree is short for complement. When we refer to a cycle, path, or spanning tree we usually actually mean the edge set of these graphs. Notice that if $e = xy$ is an edge of a cotree $\overline{T}$ of a simple graph G (so e is not an edge of T), then by Proposition 4.1 there is a unique xy-path in T, denoted $P = xTy$ (in the notation of Diestel introduced in Section 4.1). Then $T + e$ contains a unique cycle.

Definition. For a connected graph G with spanning tree T and e an edge of the cotree $\overline{T}$, the cycle described in Note 4.2.A is a *fundamental cycle* of G with respect to T and e, denoted C_e (even though the notation does not reflect the role of T).

Note. In Figure 4.4(a) spanning tree T of the wheel W_4 is given where $T = \{1, 2, 4, 5\}$. Notice the cotree $\overline{T}$ is $\{3, 6, 7, 8\}$ (we give the edge sets of spanning trees since the vertex set is implied). So W_4 has four fundamental cycles, C_3, C_6,

C_7, and C_8, as given in Figure 4.4(b).

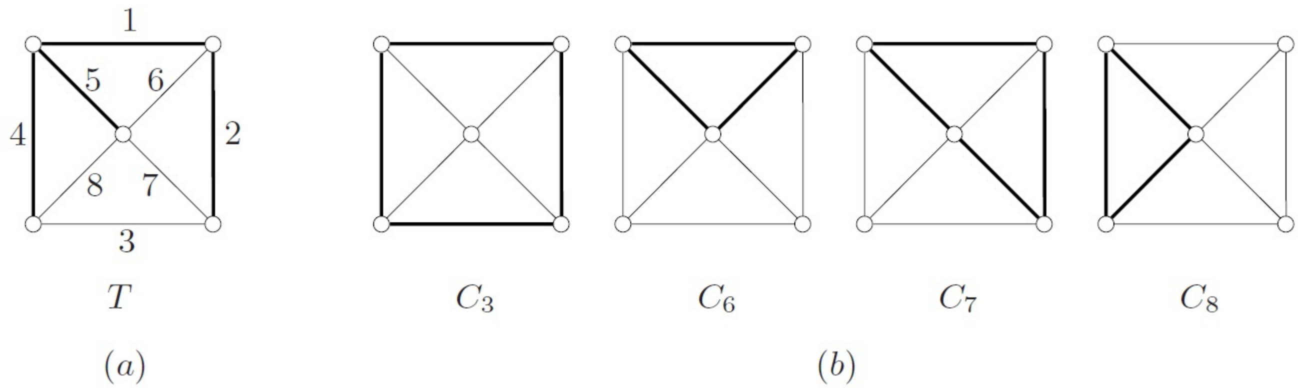

Figure 4.4. A spanning tree T and its four fundamental cycles.

Note. The properties of the fundamental cycles of a graph with respect to a spanning tree imply certain properties of the structure of the graph.

Theorem 4.10. Let T be a spanning tree of a connected graph G, and let S be a subset of its cotree $\overline{T}$. Then the symmetric difference of the fundamental cycles $C = \triangle\{C_e \mid e \in S\}$ is an even subgraph of G. Moreover, $C \cap \overline{T} = S$, and C is the only even subgraph of G with this property.

Corollary 4.11. Let T be a spanning tree of a connected graph G. Every even subgraph of G can be expressed uniquely as a symmetric difference of fundamental cycles with respect to T.

Corollary 4.12. Every cotree of a connected graph (that is, every complement of a spanning tree) is contained in a unique even subgraph of the connected graph.

Note. Two of the exercises follow from Corollary 4.12:

- **Exercise 4.3.9.** A graph which contains a Hamilton cycle has a covering by two even subgraphs.
- **Exercise 4.3.10.** A graph which contains two edge-disjoint spanning trees has (a) an Eulerian spanning subgraph, and (b) a covering by two even subgraphs.

Note. We now consider relationships between spanning trees and edge cuts. In so doing, we will get results analogous to the above results, but with even subgraphs replaced with edge cuts and fundamental cycles replaced with "fundamental bonds."

Note 4.3.B. Let G be a connected graph with spanning tree T. Since T is connected, then every nonempty edge cut of G contains at least one edge of T. So, just as the only even subgraph of T is the empty even subgraph, the only edge cut contained in cotree $\overline{T}$ is the empty edge cut. We now address fundamental bonds.

Note. Let $e = xy$ be an edge in a spanning tree T of a graph G. Then $T \setminus e$ has two components. Let X be the (vertex set of) the component containing x and let Y be the component containing y (so that $X \cup Y = V(G)$). Notice that by Note 2.5.A we have $\partial(X) = \partial(V \setminus X) = \partial(Y)$ so in terms of edge cuts, the roles of X and Y are interchangeable (so the roles played by the ends of edge e are interchangeable here). Now the two subgraphs of G induced by vertex sets X and Y are both connected so, by Theorem 2.15, $B_e = \partial(X)$ is a bond. Also, $B_e \subseteq \overline{T} \cup \{e\}$ and B_e includes e.

Lemma 4.3.A. Let G be a connected graph, T a spanning tree in G, and $e = xy$ an edge of T. Let X be the vertex set of the component of $T \setminus e$ which contains x. Then the bond $B_e = \partial(X)$ is the only bond of G contained in $\overline{T} \cup \{e\}$ that includes edge e.

Definition. Let G be a connected graph, T a spanning tree in G, and e an edge of T. Let X be the vertex set of one of the components of $T \setminus e$. Then the bond $B_e = \partial(X)$ which is contained in $\overline{T} \cup \{e\}$ that includes e is the *fundamental bond* of G with respect to T and e.

Note. In Figure 4.5, a spanning tree T in the wheel W_4 is given (in Figure 4.5(a)), along with the four fundamental bonds with respect to T (in Figure 4.5(b)).

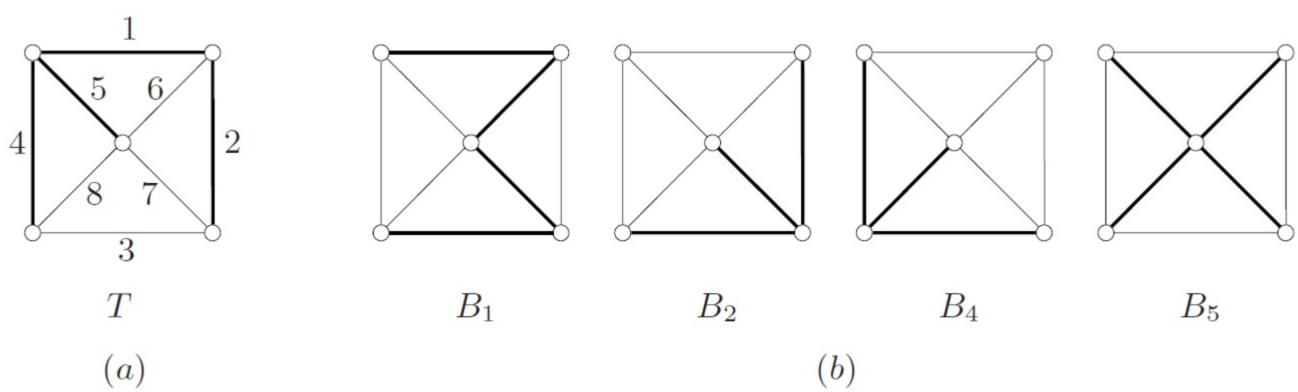

Figure 4.5

Note. The following three results are analogous to Theorem 4.10, Corollary 4.11, and Corollary 4.12, respectively. The proofs are to be given in Exercise 4.3.5.

Theorem 4.13. Let T be a spanning tree of a connected graph G, and let S be a subset of T. Set $B = \triangle\{B_e \mid e \in S\}$. Then B is an edge cut of G. Moreover, $B \cap T = S$ and B is the only edge cut of G with this property.

Corollary 4.14. Let T be a spanning tree of a connected graph G. Every edge cut of G can be expressed uniquely as a symmetric difference of fundamental bonds with respect to T.

Corollary 4.15. Every spanning tree of a connected graph is contained in a unique edge cut of the graph.

Note 4.3.C. In Exercise 4.3.6(a) it is to be shown that the fundamental cycles of a connected graph G (with respect to a given spanning tree T) form a basis for the cycle space of the graph, and the fundamental bonds of G (with respect to given spanning tree T) form a basis of its bond space.

Definition. The dimension of the cycle space of a graph is the graph's *cyclomatic number*.

Note 4.3.D. By Exercise 4.3.6(b), we see that the cyclomatic number of a connected graph G is the number of fundamental cycles of G with respect to any given spanning tree T (namely, $m - n + 1$). It is also to be shown that the number of fundamental bonds of a connected graph G (and so the dimension of the bond space $\mathcal{B}(G)$) is $n - 1$.

Note 4.3.E. We now summarize what we know about the edge space, cycle space, and bond space of a connected graph.

Space	**Vectors**	**spanning set**	**basis**	**dimension**
Cycle Space	even subgraphs	cycles	fundamental cycles	$m - n + 1$
Bond Space	edge cuts	bonds	fundamental bonds	$n - 1$
Edge Space	sets of edges	cycles & bonds	fundamental cycles & fundamental bonds	m

We also know a bit about the geometry of these spaces. In Exercise 2.6.4(c) it is to be shown that the cycle space and the bond space are orthogonal complements of each other in the edge space of G, which justifies the fact that the dimension of the edge space is $(m - n + 1) + (n - 1) = m$.

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