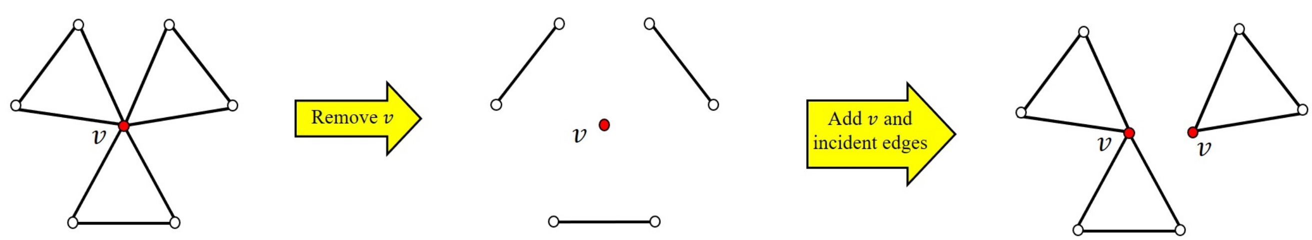
Section 5.2. Separations and Blocks

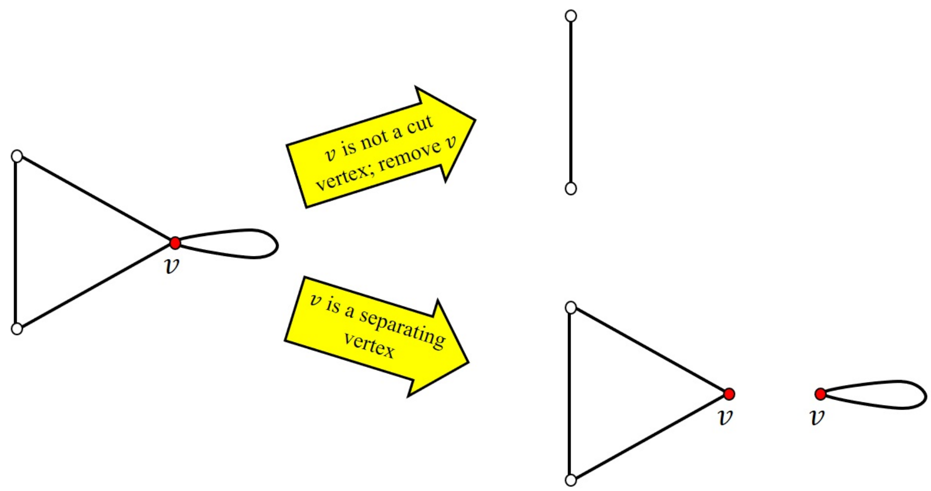
Note. In this section we introduce two ideas closely related to the concepts of a cut vertex and a connected component. Some properties are given and the Double Cover Conjecture is revisited.

Definition. A *separation* of a connected graph is a (edge) decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. The common vertex is a *separating vertex* of the graph. The *separating vertices of a disconnected graph* are the separating vertices of its components.

Note 5.2.A. A cut vertex of a graph is a separating vertex since we can take one of the new components that result from the removal of the cut vertex, add back the cut vertex and all edges between it and the one new component, *plus* take the remaining new components and add back the cut vertex and all edges between it and the remaining new components, to produce the desired partitioning:



However, a separating vertex need not be a cut vertex. Consider a vertex with a loop and at least one other edge. It may not be a cut vertex (depending on its neighbors) but it is a separating vertex since one of the parts of the partition in the definition of "separation" is the loop and the separating vertex:



In a loopless graph, a separating vertex yields a decomposition of the graph into nonempty subgraphs from which the separating vertex can be removed yielding more components of $G - v$ than there are of G. That is, in a loopless graph, the cut vertices and the separating vertices are identical. Figures 5.1 and 5.3 give the cut vertices and the separating vertices for a particular graph:

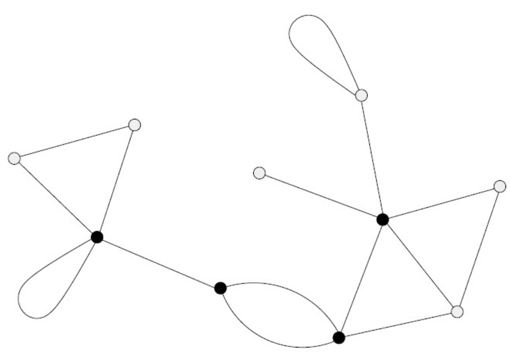


Fig. 5.1. The cut vertices of a graph

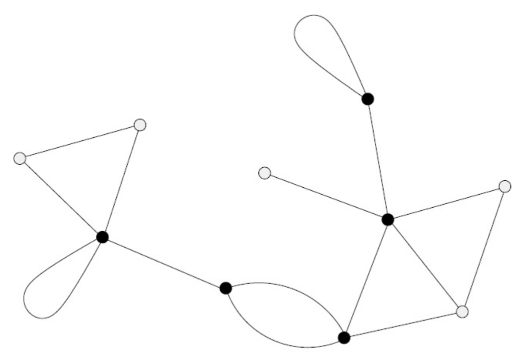


Fig. 5.3. The separating vertices of a graph

Definition. A graph is *nonseparable* if it is connected and has no separating vertices. A graph that is not nonseparable is said to be *separable* (whether it is connected or not).

Note 5.2.B. There are two nonseparable graphs on one vertex, namely K_1 and K_1 with a loop. (A single vertex with multiple loops can be decomposed into single loops in the sense of the use of the term in the definition of "separation.") All connected graphs on two or more vertices without separating vertices are loopless (as described above). Since a separation only involves the behavior of a vertex and connectivity (through the existence of paths, say), then multiple edges are irrelevant so that a loopless graph is nonseparable if and only if its underlying simple graph is nonseparable.

Note. Cycles are nonseparable graphs. Cycles are fundamental to the nonseparability of a graph as spelled out by H. Whitney in 1932 with the following.

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Note. We now address the second topic of this section, that of a "block." We will use the blocks of a graph to give a "sort-of" partition of the graph. Many properties of a connected graph can be addressed by considering properties of the blocks that make up the graph.

Definition. A *block* of a graph is a subgraph which is nonseparable and is maximal with respect to this property.

Note. Notice that a nonseparable graph has just one block (the graph itself). The blocks of a (nontrivial) tree are the copies of K_2 induced by its edges because every vertex of tree of degree greater than 1 is a cut vertex and hence a separating vertex. The separating vertices of Figure 5.3 produce the blocks of Figure 5.4(a):

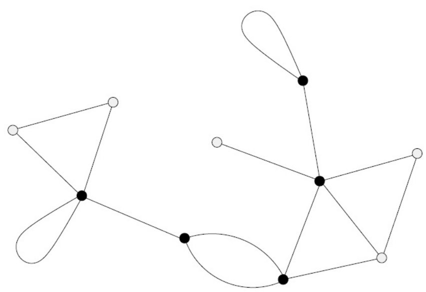


Fig. 5.3. The separating vertices of a graph

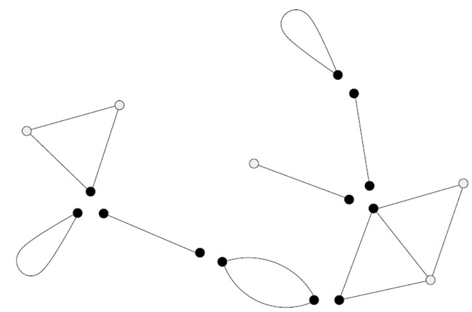


Fig. 5.4. (a) The blocks of the graph of Figure 5.3

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Definition. For any graph G, define the graph $B(G)$ with vertex set $\mathcal{B} \cup S$ where $\mathcal{B}$ is the set of blocks of G, and S is the set of separating vertices. In the edge set of $B(G)$ we have block $B \in \mathcal{B}(G)$ and separating vertex $v \in S$ adjacent if and only if B contains v. For connected graph G, graph $B(G)$ is the *block tree* of G (we'll see below that $B(G)$ actually is a tree when G is connected). If G is separable, the blocks of G which correspond to leaves of $B(G)$ are *end blocks*. An *internal vertex* of a block of a graph G is a vertex which is not a separating vertex of G.

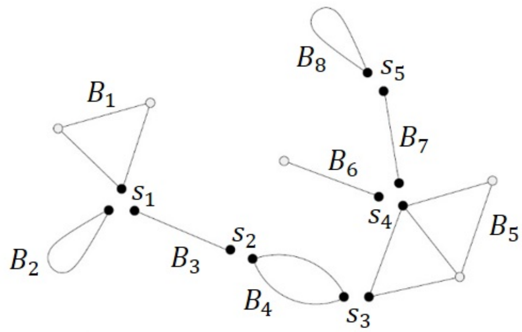
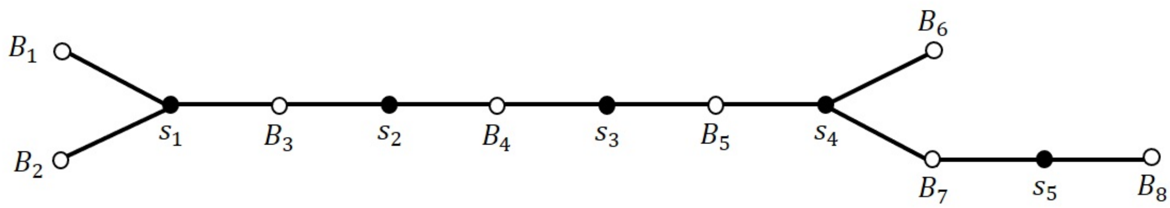


Fig. 5.4. (a) The blocks of the graph of Figure 5.3



The block tree of based on the blocks given in Figure 5.3

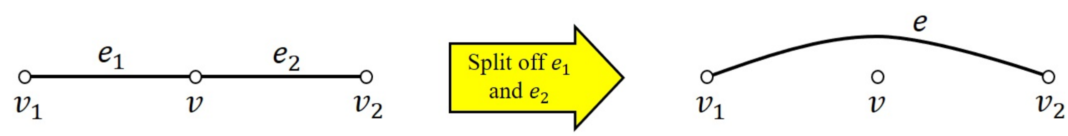
Note 5.2.C. The block tree $B(G)$ is bipartite with bipartition $(\mathcal{B}, S)$. Now a path in G connecting vertices of G in distinct blocks (and so the path must pass though some separating vertices of G) "gives rise to" a unique path in $B(G)$ connecting these same blocks (the uniqueness follows from the fact that any cycles in G are contained within a single block of G by Proposition 5.3(c)). So if G is connected then $B(G)$ is connected. Also by Proposition 5.3(c), $B(G)$ is acyclic. So if G is connected then $B(G)$ is a tree.

Note. Many properties of a graph can be addressed by considering the same properties of the blocks of the graph. For example, in Exercise 5.2.5 it is to be shown that a graph is even if and only if each of its blocks is even, and a graph is bipartite if and only if each of its blocks is bipartite. Notice that these claims do not assume that the given graph is connected, but we can consider the connected components of the graph and then the block tree of each component. In Exercise 5.2.8 it is to be shown that a spanning subgraph T of a connected G is a spanning tree of G if and only if $T \cap B$ is a spanning tree of B for every block B of G. Similarly, by Proposition 5.3 (b and c) a graph has a cycle double cover if and only if each of its blocks has a cycle double cover. So the Cycle Double Cover Conjecture can be reduced to considering nonseparable graphs (since blocks are, by definition, nonseparable). We'll see below in Theorem 5.5 that this idea can be further refined.

Note. We now define a procedure that will be useful in addressing cycle decompositions and cycle coverings.

Definition. Let v be a vertex of a graph G and let $e_1 = vv_1$ and $e_2 = vv_2$ be two edges of G incident to v. The operation of *splitting off* the edges e_1 and e_2 from v consists of deleting e_1 and e_2 and then adding a new edge e joining v_1 and v_2. (If $v_1 = v_2$ then e_1 and e_2 are parallel edges and splitting off e_1 and e_2 amounts to replacing e_1 and e_2 with a loop on vertex $v_1 = v_2$.)

Note. Splitting off edges e_1 and e_2 can be illustrated as:



In the next result, some conditions are given under which splitting off edges can be performed without creating cut edges.

Theorem 5.4. The Splitting Lemma.

Let G be a nonseparable graph and let v be a vertex of G of degree at least four with at least two distinct neighbors. Then some two nonparallel edges incident to v can be split off so that the resulting graph is connected and has no cut edges.

Note. We observed above that we can address the Cycle Double Cover Conjecture by considering nonseparable graphs. We can use the Splitting Lemma to show that we can restrict ourselves to an even smaller collection of graphs.

Theorem 5.5. The Cycle Double Cover Conjecture is true if and only if it is true for all nonseparable cubic graphs.

Note. In Veblen's Theorem (Theorem 2.7) we saw that a graph has a cycle decomposition if and only if the graph is even. In Exercise 5.2.12 additional properties of cycle decompositions are given. The Splitting Lemma is to be used to show that every even graph has an odd number of cycle decompositions, and that each edge of an even graph lies in an odd number of cycles.

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