Section 5.3. Ear Decompositions

Note. In this section we give a technique to construct a nonseparable graph from a cycle and paths. The construction is based on an "ear decomposition." The recursive nature of the construction allows us to give to address some properties of nonseparable graphs with mathematical induction. This is illustrated in Theorem 5.10 in which we show that every connected graph without cut edges has a strong orientation.

Definition. Let F be a subgraph of a graph G. An *ear* of F in G is a nontrivial path in G whose ends lie in F but whose internal vertices do not.

Note. The term "ear" is chosen for obvious anatomical analogies! Some ears of nonseparable graphs are given in the next result.

Proposition 5.6. Let F be a nontrivial proper subgraph of a nonseparable graph G. Then there is an ear of F in G.

Note. The next result lets us build larger nonseparable subgraphs of a graph G using proper nonseparable subgraph F of G and an ear of F in G. The proof is to be given in Exercise 5.3.1.

Proposition 5.7. Let F be a nonseparable proper subgraph of a graph G, and let P be an ear of F in G. Then $F \cup P$ is nonseparable.

Definition. A *nested sequence* of graphs is a sequence $(G_0, G_1, \ldots, G_k)$ of graphs such that $G_i \subset G_{i+1}$ for $0 \leq i < k$. An *ear decomposition* of a nonseparable graph G is a nested sequence $(G_0, G_1, \ldots, G_k)$ of nonseparable subgraphs of G such that

▷ G_0 is a cycle,

▷ $G_{i+1} = G_1 \cup P_i$ where P_i is an ear of G_i in G for $0 \leq i < k$,

▷ $G_k = G$.

Note. An ear decomposition of the Petersen graph is given in Figure 5.6 below. The edges of the initial cycle (all of G_0) and the edges of the ear added at each stage are given in darker lines.

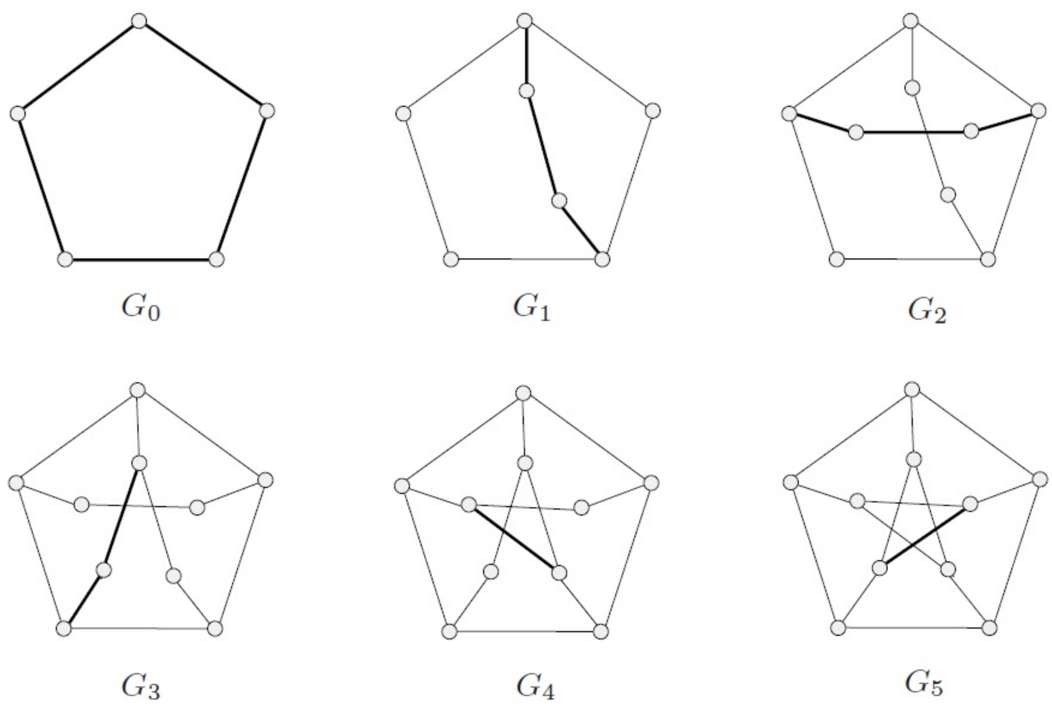
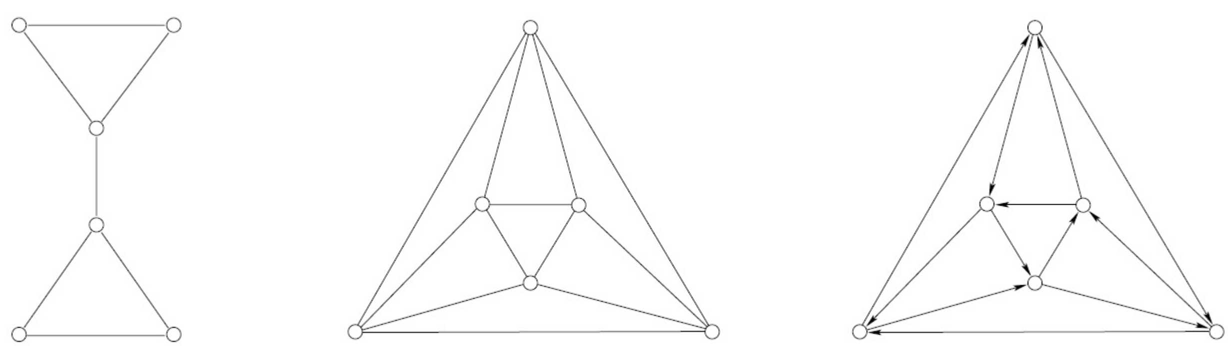


Figure 5.6. An ear decomposition of the Petersen graph

Theorem 5.8. Every nonseparable graph other then K_1 and K_2 has an ear decomposition.

Note. Recall that a digraph D is strongly connected if the outcuts satisfy $\partial^+(X) \neq \varnothing$ for every nonempty proper subset C of $V(D)$. We will classify the class of connected graphs which have orientations that make a strongly connected digraph.

Note. Suppose a graph represents a network of roads between locations. We want to consider the problem of making each location from any other location. That is, we want to put an orientation on the graph so that the resulting digraph is strongly connected. In Figure 5.7(a), a graph is given that does not have such an orientation because the graph has a cut edge. Clearly no graph with a cut edge can be so oriented. In fact, we show below that any connected cut-edge-free graph has a "strong orientation." The proof is based on considering the blocks of the graph. Figure 5.7(b and c) gives an example of an edge-cut-free graph and its strong orientation.



(a) (b) (c)

Figure 5.7.

Note. The next result is another property of a graph which can be addressed by considering the blocks of the graph. The proof is to be given in Exercise 5.3.9.

Proposition 5.9. A connected digraph is strongly connected if and only if each of its blocks is strongly connected.

Note. Since no graph with a cut edge has a strong orientation, the next theorem can actually stated as an if and only if result.

Theorem 5.10. Every connected graph without cut edges has a strong orientation.

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