Chapter 9. Connectivity

Section 9.1. Vertex Connectivity

Note. Consider the graphs on five vertices in Figure 9.1.

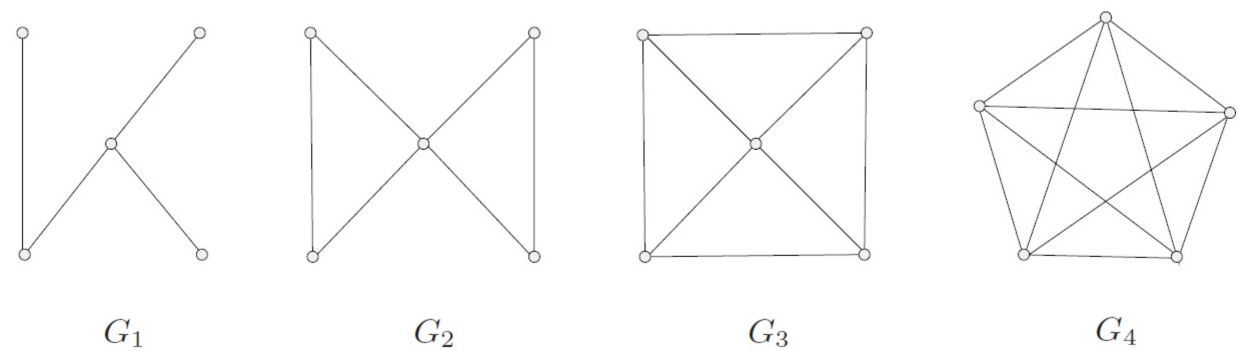
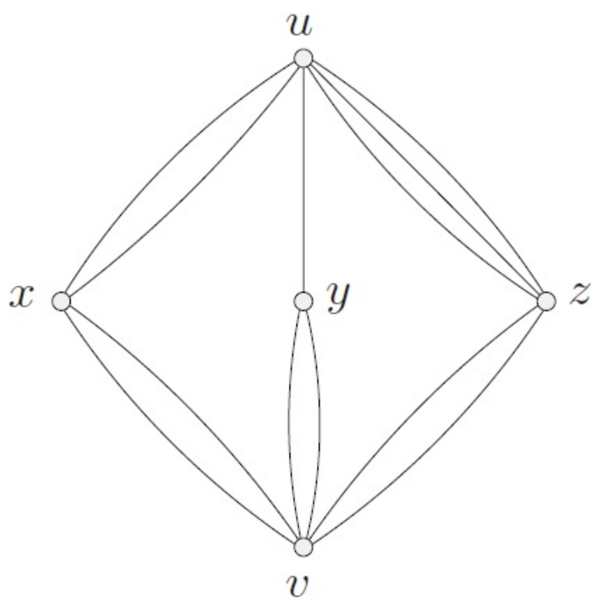


Figure 9.1

Each is connected, but to different degrees. Graph G_1 is a tree so that the deletion of any edge produces a disconnected graph. Graph G_2 cannot be disconnected by the deletion of one edge, but can be disconnected by the deletion of one vertex (the cut-vertex of degree 4). Graph G_3 has neither cut edges nor cut vertices, as is the case for graph G_4. Our intuition tells us that $G_4 = K_5$ is more connected than G_3. We introduce two connectivity parameters, one related to vertices and one related to edges.

Definition. Let x and y be distinct vertices in graph G. The *local connectivity* between x and y is the maximum number of pairwise internally disjoint xy-paths, denoted $p(x, y)$. (Recall that two xy-paths are internally disjoint if the only vertices they share are x and y.)

Note. We can put the local connectivities of a graph in a matrix where the entries are indexed by pairs of vertices. Figures 9.2(a,b) gives a graph and the matrix of local connectivities (notice that we leave $p(x,x)$ undefined).



(a)

	u	x	y	z	v
u	$*$	3	2	4	3
x	3	$*$	2	2	3
y	2	2	$*$	2	3
z	4	2	2	$*$	3
v	3	3	3	3	$*$

(b)

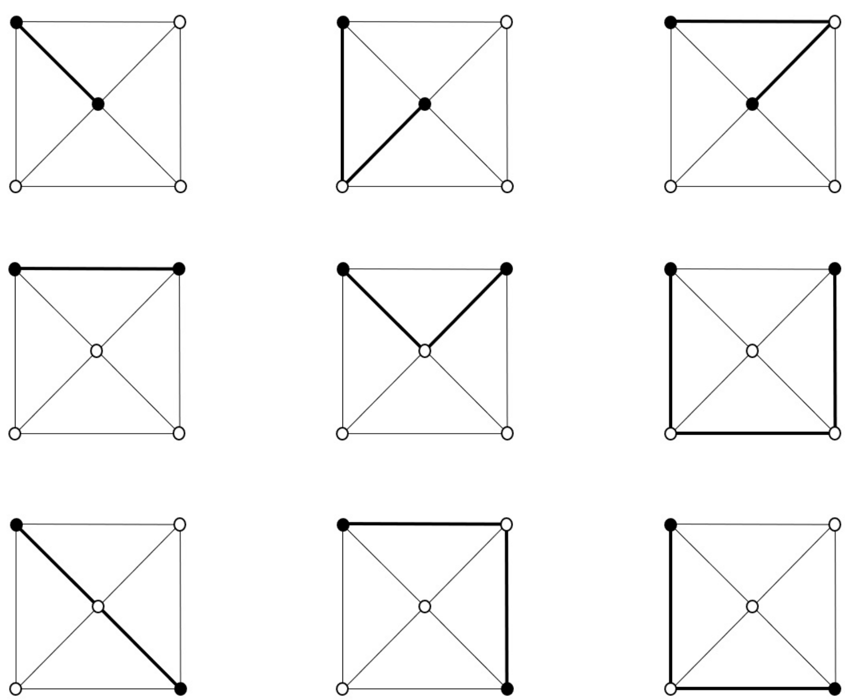
Figure 9.2(a,b).

Definition. A nontrivial graph G is *k-connected* if $p(u,v) \geq k$ for every pair of distinct vertices u and v of G. We take (by definition) the trivial graphs to be both 0-connected and 1-connected, but not k connected for $k > 1$. The *connectivity* of G, $\kappa(G)$, is the maximum value of k for which G is k-connected (and so the connectivity of G is the minimum value of $p(u,v)$ over all distinct vertices u and v of G). That is, for nontrivial graph G,

$$\kappa(G) = \min\{p(u,v) \mid u,v \in V(G), u \neq v\}.$$

Note. It is common to define the connectivity $\kappa(G)$ as the minimum size of a vertex cut. This is the approach taken in Bondy and Murty's *Graph Theory with Applications*, North-Holland 1976 (see their Section 3.1. Connectivity) and in Béla Bollabás's *Modern Graph Theory*, Springer 2002 (see his Section III.2. Connectivity and Menger's Theorem). The equivalence of these definitions is given by Menger's Theorem (see Theorem 9.1 and Note 9.1.C below).

Note. By Exercise 3.1.4, graph G is connected if and only if for any distinct vertices u and v of G, G contains a uv-path. Therefore, a graph is connected if and only if it is 1-connected. Equivalently, a nontrivial graph G is disconnected if it has connectivity $\kappa(G) = 0$. The graphs in Figure 9.1 above satisfy $\kappa(G_1) = 1$, $\kappa(G_2) = 1$, $\kappa(G_3) = 3$, and $\kappa(G_4) = 4$. For example, in G_3 we have three internally disjoint paths between different "types" of vertices as follows:



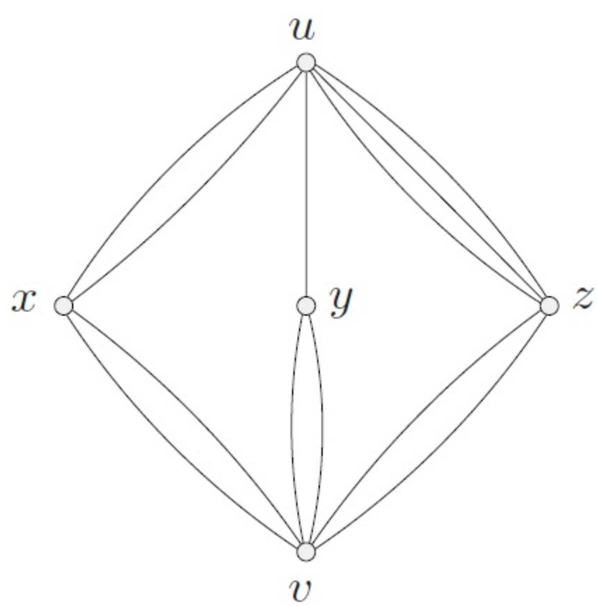
The graph in Figure 9.2(a) is 1-connected and 2-connected, but is not 3-connected (because there are only two internally disjoint paths between u and y, say) so that its connectivity is 2.

Note 9.1.A. Consider K_n where $n \geq 2$. For x and y vertices of K_n, there is one path of length one connecting x and y (which has *no* internal vertices), and there are $n-2$ paths of length two connecting x and y (one through each of the $n-2$ vertices of K_n except for x and y). These $n-1$ paths are internally disjoint (and there can be no more internally disjoint paths from these) so $p(x,y) = n-1$. This is true for any pair of vertices in K_n, so $\kappa(K_n) = n-1$ where $n \geq 2$.

Note 9.1.B. In computing the local connectivity $p(x,y)$, we are interested in internally disjoint paths so, except in the case of paths of length one, parallel edges do not affect $p(x,y)$. So, except when considering paths of length one, we may as well consider the underlying simple graph of a graph G. If the underlying simple graph of graph G is complete and x and y are joined by $\mu(x,y)$ links (i.e., nonloops), then there are $\mu(x,y)$ paths of length one joining x and y and, as above, $n-2$ internally disjoint paths of length two joining x and y (and there can be no additional paths internally disjoint from these). So $p(x,y) = n-2+\mu(x,y)$. The connectivity of any nontrivial graph G whose underlying simple graph is a complete graph is $\kappa(G) = n-2+\mu$ where μ is the minimum edge multiplicity in G. Now if x and y are nonadjacent in graph H (and so are vertices of a graph whose underlying simple graph is not complete) then there are at most $n-2$ internally disjoint paths connecting x and y. In this case, the connectivity of H is at most $n-2$. Next we want to relate the connectivity of a graph to the number of vertices whose deletions result in a disconnected graph.

Definition. Let x and y be distinct nonadjacent vertices of graph G. An *xy-vertex-cut* is a subset S of $V \setminus \{x, y\}$ such that x and y belong to different components of $G - S$. In this case we say S *separates x and y*. The minimum size of a vertex cut separating x and y is denoted by the function $c(x, y)$, called the *local cut function* of G (which is undefined if $x = y$ or x and y are adjacent). A vertex cut separating some pair of nonadjacent vertices of G is a *vertex cut* of G, and one with k elements is a *k-vertex cut*.

Note. In Figure 9.2(a,c), an example of the local cut function is given in matrix form:



(a)

	u	x	y	z	v
u	$*$	$*$	$*$	$*$	3
x	$*$	$*$	2	2	$*$
y	$*$	2	$*$	2	$*$
z	$*$	2	2	$*$	$*$
v	3	$*$	$*$	$*$	$*$

(c)

Figure 9.2(a,c).

Note. A complete graph has no vertex cuts, nor does any graph whose underlying simple graph is complete. In fact, these are the only graphs with no vertex cuts since we will show in Theorem 9.2 below that if G has at least one pair of nonadjacent vertices, then the size of a minimum vertex cut of G is equal to the connectivity of G. First, we need a theorem of K. Menger from 1927. The inductive proof given below is due to F. Göring in 2000.

Definition. To *shrink* a proper subset X of vertices in graph G is to delete all edges between vertices of X and then identify the vertices of X as a single vertex. The resulting graph is denoted G/X (read "G modulo X").

Theorem 9.1. MENGER'S THEOREM (UNDIRECTED, VERTEX VERSION).
In any graph $G(x, y)$, where x and y are nonadjacent, the maximum number of pairwise internally disjoint xy-paths is equal to the minimum number of vertices in an xy-vertex-cut, that is, $p(x, y) = c(x, y)$.

Note 9.1.C. The equality in Menger's Theorem implies

$$\min\{p(u, v) \mid u, v \in V, u \neq v, uv \notin E\} = \min\{c(u, v) \mid u, v \in V, u \neq v, uv \notin E\} \qquad (9.2)$$

If G is a graph that has at least one pair of nonadjacent vertices, then the right hand side of (9.2) is the size of a minimum vertex cut of G (since $c(u, v)$ is undefined for adjacent u and v). The connectivity of G is

$$\kappa(G) = \min\{p(u, v) \mid u, v \in V, u \neq v\}$$

and notice that there is no constraint of $uv \notin E$ in the definition of κ. In the next result we show, using Menger's Theorem, that for G with at least one pair of nonadjacent vertices, κ can be computed using non-adjacent vertices only.

Theorem 9.2. If G has at least one pair of nonadjacent vertices, then

$$\kappa(G) = \min\{p(u,v) \mid u,v \in V, u \neq v, uv \notin E\}. \qquad (9.3)$$

Note. Combining Theorem 9.1 and Theorem 9.2, we see that for a graph G with at least one pair of nonadjacent vertices,

$$\kappa(G) = \min\{c(u,v) \mid u,v \in V, u \neq v, uv \notin E\}. \qquad (9.4)$$

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