Section 9.2. The Fan Lemma

Note. In this section we define a fan and prove The Fan Lemma using Menger's Theorem (Theorem 9.1). We then use it to prove a result concerning the existence of certain cycles in a k-connected graph (see Theorem 9.6).

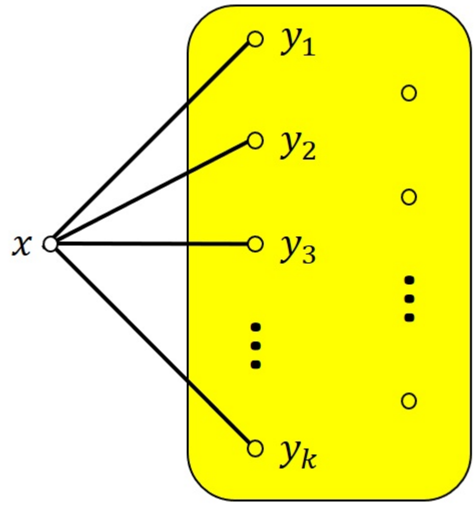
Lemma 9.3. Let G be a k-connected graph and let H be a graph obtained from G by adding a new vertex y and joining it to at least k vertices of G. Then H is also k-connected.

Note. In Section 9.1 we related internally disjoint xy-paths to connectivity (see Theorem 9.2). The next result involves the existence of (X, Y)-paths in a k-connected graph.

Proposition 9.4. Let G be a k-connected graph, and let X and Y be subsets of V of cardinality at least k. Then there exists in G a family of k pairwise disjoint (X, Y)-paths.

Note. A family of k internally disjoint (X, Y)-paths whose terminal vertices are distinct is a k-*fan* from x to Y.

Note. The term "fan" is appropriate:



Note. The proof of The Fan Lemma (the statement is given next) is similar to the proof of Proposition 9.4 and is to be given in Exercise 9.2.1.

Proposition 9.5. The Fan Lemma.
Let G be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y.

Note 9.2.A. Recall that Theorem 5.1 states: "A connected graph on three or more vertices has no cut vertices if and only if any two distinct vertices are connected by two internally disjoint paths." So in a 2-connected graph (such a graph G has no cut vertices and so any xy-vertex cut of G consists of at least 2 vertices; $c(x, y) \geq 2$ for all $x, y \in G$ so that $\kappa(G) \geq 2$), and two vertices are connected by two internally disjoint paths. Since these two paths together form a cycle, then we have that in a 2-connected graph, any two vertices lie on a common cycle. G. A. Dirac ("Gabriel Andrew Dirac," not to be confused with the physicist Paul A. M. Dirac) generalized this idea of vertices lying on a common cycle and proved the following for k-connected graphs in 1952.

Theorem 9.6. Let S be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all vertices of S.

Note. In Theorem 9.6 the ("cyclic") order in which the vertices of set S occur on the cycle is not specified. To show that in general no specific order can be given, consider Figure 9.6 which gives a 4-connected graph with vertices x_1, y_1, x_2, y_2. There is no cycle including these four vertices in the order given because every x_1y_1-path intersects every x_2y_2-path. Exercise 9.2.4 also deals with vertices appearing on a cycle in a given order.

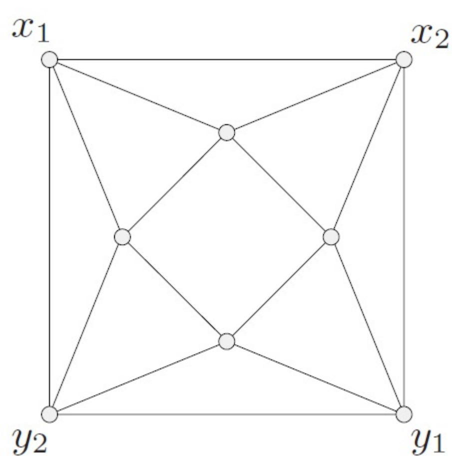


Figure 9.2.