Section 9.3. Edge Connectivity

Note. We now shift our attention from vertex connectivity to edge connectivity. We give several definitions and some new versions of Menger's Theorem, but no proofs are presented.

Definition. The *local edge connectivity* between distinct vertices x and y is the maximum number of pairwise edge-disjoint xy-paths, denoted $p'(x, y)$. Local edge connectivity is undefined when $x = y$. A nontrivial graph G is *k-edge-connected* if $p'(u, v) \geq k$ for any two distinct vertices u and v of G. We take the trivial graph to be both 0-edge-connected and 1-edge-connected, but not k-edge-connected for any $k > 1$. The *edge connectivity* of graph G is the maximum value of k for which G is k-edge-connected, denoted $\kappa'(G)$. That is, $\kappa'(G) = \min\{p'(u, v) \mid u, v \in V(G), u \neq v\}$.

Note. As with vertex connectivity, it is common to define edge connectivity in terms of the smallest edge cut of the graph. This is the approach taken in Bondy and Murty's *Graph Theory with Applications*, North-Holland 1976 (see their Section 3.1. Connectivity) and in Béla Bollabás's *Modern Graph Theory*, Springer 2002 (see his Section III.2. Connectivity and Menger's Theorem). The equivalence of these definitions is given by Menger's Theorem (see Theorem 9.7 and Note 9.3.A below).

Note. A graph is connected if and only if it is 1-edge connected (not to be confused with edge connectivity 1). A graph is disconnected if and only if its edge connectivity is 0. Consider again the graphs in Figure 9.1. They satisfy $\kappa'(G_i) = i$ for $i \in \{1, 2, 3, 4\}$.

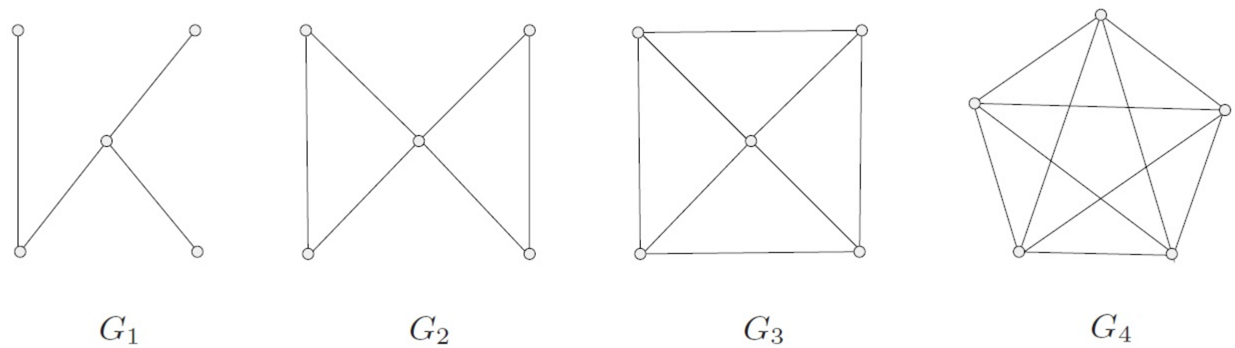


Figure 9.1

Note. Recall that edge cut $\partial(X)$ is the set of edges of G with one end in X and one end in $V \setminus X$. For distinct vertices x and y in graph G, recall that edge cut $\partial(X)$ "separates" x and y if $x \in X$ and $y \in V \setminus X$. We denote the minimum cardinality of an edge cut separating x and y as $c'(x, y)$. We can now state Menger's Theorem in terms of c' and local edge connectivity p'. This was originally stated as Theorem 7.17 and a proof was given based on flows. In Exercise 9.3.12 a proof is to be given using Menger's Theorem (undirected Vertex Version; Theorem 9.1).

Theorem 9.7. Menger's Theorem (Edge Version).

For any graph $G(x, y)$, we have $p'(x, y) = c'(x, y)$.

Definition. A *k-edge-cut* is an edge cut $\partial(X)$, where $\varnothing \subsetneq X \subsetneq V$ and $|\partial(X)| = k$.

Note 9.3.A. Since $\varnothing$ is a proper subset of X, and X is a proper subset of V, then $X \neq \varnothing$ and $V \setminus X \neq \varnothing$. So a k-edge-cut separates some pair of vertices. So by the definition of edge connectivity $\kappa'(G)$ and Menger's Theorem (Edge Version; Theorem 9.7), we have that (for nontrivial graph G) $\kappa'(G)$ is equal to the least integer k for which G has a k-edge cut.

Note. We can relate the connectivity κ, edge connectivity κ', and minimum degree δ of a graph G as $\kappa \leq \kappa' \leq \delta$, as is to be shown in Exercise 9.3.2(a). In Exercise 9.3.2(b) it is to be shown by example that each of the inequalities can be strict. An example for which $\kappa = 2$, $\kappa' = 3$, and $\delta = 4$ is given in Bondy and Murty's *Graph Theory with Applications*:

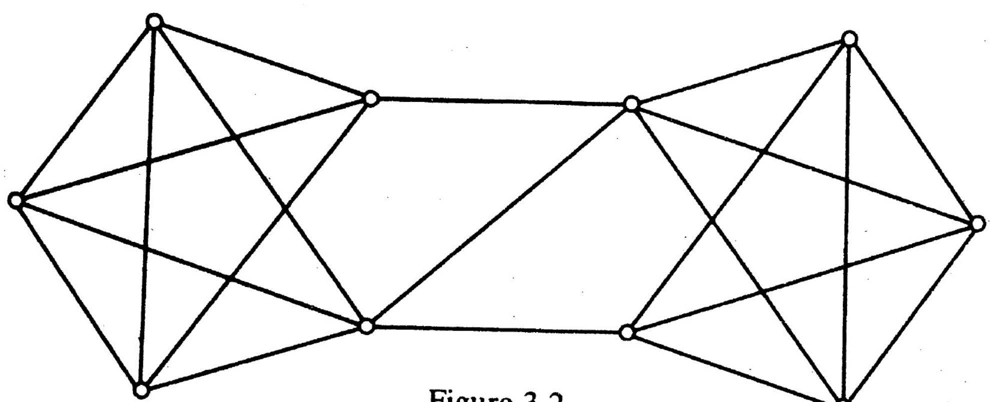
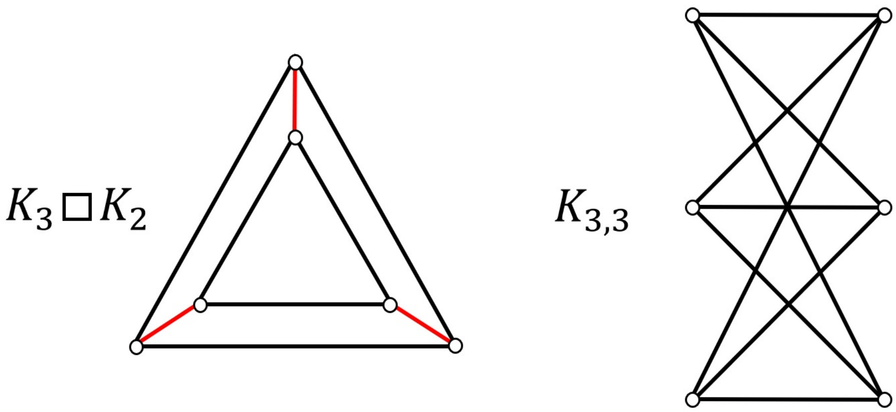


Figure 3.2

Definition. A k-edge-connected graph is *essentially* $(k+1)$*-edge connected* if all its k-edge cuts are trivial (recall that a trivial edge cut is one associated with a single vertex and so is the collection of edges incident to a single vertex).

Note 9.3.B. The graphs $K_{3,3}$ and $K_3 \square K_2$ are both 3-regular (so $\delta = 3$), with connectivity $\kappa = 3$, and with edge connectivity $\kappa' = 3$. However, $K_{3,3}$ is essentially 4-edge-connected, but $K_3 \square K_2$ is not essentially 4-edge connected because it has the nontrivial 3-edge cut given below:



Note. It is to be shown in Exercise 9.3.8 that if G is a k-edge-connected graph and if $\partial(X)$ is a k-edge cut, then the graphs G/X and $G/\overline{X}$ (obtained by shrinking X to a single vertex x and shrinking $\overline{X} = V \setminus X$ to a single vertex $\overline{x}$, respectively) are also k-edge connected. Iterating this shrinking process one can take a k-edge-connected graph (with $k \geq 1$) and "decompose" it into a collection of essentially $(k+1)$-edge-connected graphs. "For many problems," one can just consider the "components" individually (see page 217).

Note. We can also consider edge connectivity in digraphs. Since edge connectivity is defined in terms of paths, we just replace "path" with "directed path" in the digraph setting.

Definition. A (x, y)-*vertex-cut* in a digraph is a subset S of $V \setminus \{x, y\}$ whose deletion destroys all directed (x, y)-paths.

Note. We can now state another version of Menger's Theorem, this time a vertex version for digraphs.

Theorem 9.8. Menger's Theorem (Directed Vertex Version).
In any digraph $D(x, y)$, where are $(x, y) \notin A(D)$, the maximum number of pairwise internally disjoint directed (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex-cut.

Note. We have seen three versions of Menger's Theorem in this chapter: the undirected vertex version (Theorem 9.1), the individual edge version (Theorem 9.7), and the directed vertex version (Theorem 9.8). This just leaves the directed arc version, which was stated on Theorem 7.16 but which we repeat here.

Theorem 7.16. Menger's Theorem (Arc Version).
In any digraph $D(x, y)$, the maximum number of pairwise arc-disjoint directed (x, y)-paths in equal to the minimum number of arcs in an (x, y)-cut.

Revised: 2/13/2023