Section 9.4. Three-Connected Graphs

Note. We consider certain types of decompositions of 2-connected graphs into 3-connected components. We then consider an application related to The Double Cycle Cover Conjecture, and prove a result used in our study of planar graphs in the next chapter.

Note. To review, a graph G is 3-connected if for every pair of distinct vertices u and v in G, there are at least three internally disjoint paths between u and v. By convention, **in this section we only consider loopless graphs**.

Definition. Let G be a connected graph which is not complete, let S be a vertex cut of G, let X be the vertex set of a component of $G - S$. The subgraph H of G induced by $S \cup X$ is an *S-component* of G.

Definition. If G is 2-connected and $S = \{x, y\}$ is a 2-vertex cut of G, then we modify each S-component by adding a new edge between x and y. This edge between x and y is a *marker edge* and the modified S-components are *marked S-components.* The set of marked S-components constitute the *marked S-decomposition* of G.

Note. The original graph G can be recovered from its marked X-decomposition by unioning the marked S-components and then deleting the marker edge. This is illustrated in Figure 9.7 where the 2-vertex cut S is given by solid dots (top), the marked S-components are given (middle), and the recovery of G is given (bottom).

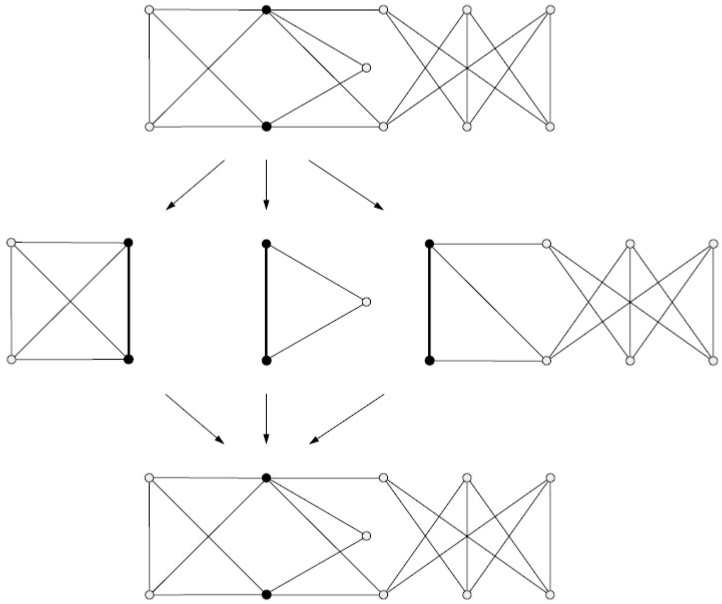

Figure 9.7

Theorem 9.9. Let G be a 2-connected graph and let S be a 2-vertex cut of G. Then the marked S-components of G are also 2-connected.

Note/Definition. By Theorem 9.9, a 2-connected graph G with a 2-vertex cut S can be decomposed into marked S-components that are also 2-connected. If a marked S-component itself has a 2-vertex cut then this component can be further decomposed by Theorem 9.9. So by iterating this process, G can be decomposed into 2-connected graphs which do not have 2-vertex cuts (with the caveat that the components are not connected components, but instead are marked X-components for some set S of two vertices of G). This allows us to create a *decomposition tree* of G reflecting the repeated applications of Theorem 9.9. This is more like a family tree than a graph theory tree! The "last descendants" in the decomposition tree are the marked S-components which do not have 2-vertex cuts (so the last descendants, or *leaves* of the decomposition tree, are either 3-connected or are graphs whose underlying graph is a complete graph on 3 vertices; notice that K_3 has connectivity 2 by Note 9.1.A but that it does not have a vertex cut of size 2 because Theorem 9.2 and equation (9.4) do not apply to complete graphs). The 3-connected leaves of the decomposition tree are called 3-*connected components* of G. Graph G is the *root* of the decomposition tree. In Figure 9.8, the decomposition tree of a 2-connected graph with a 2-vertex cut is given.

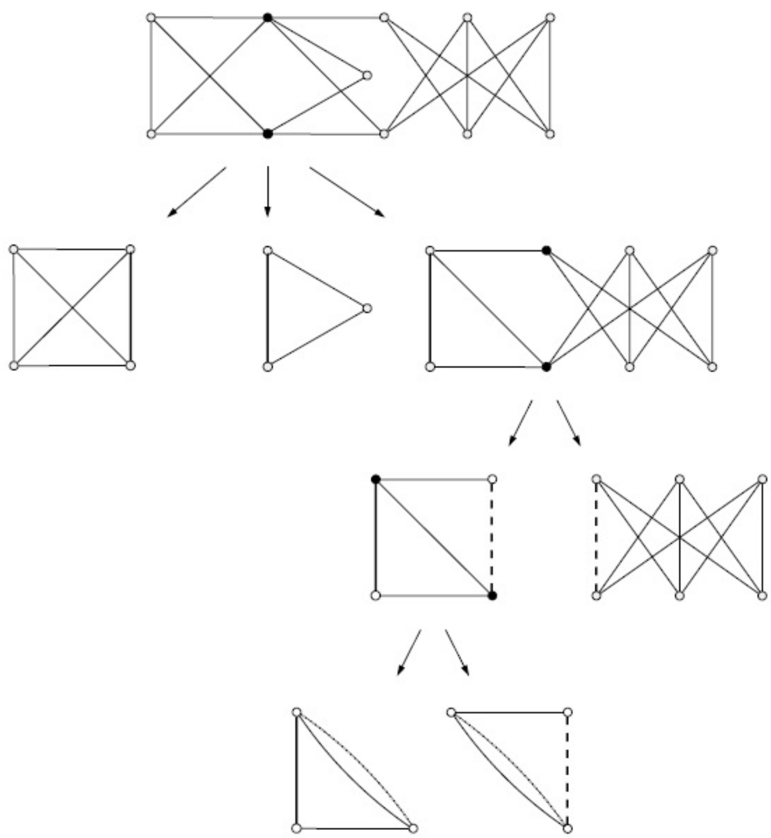

Figure 9.8

Note 9.4.A. In the process of creating the decomposition tree of G, there may be multiple choices for a 2-vertex cut. In Exercise 9.4.1 it is to be shown by example that different leaves may result from different choices of 2-vertex sets. However, it was shown by W. H. Cunningham and J. Edmonds in "A Combinatorial Decomposition Theory," *Canadian Journal of Mathematics*, **32**, 734–765 (1980) that any two applications of the procedure always result in the same set of 3-connected components (but possibly with different edge multiplicities). Cunningham and Edmonds paper is available online on the Canadian Journal of Mathematics webpage (accessed 1/10/2021).

Note. We can use the idea of a decomposition tree to gain insight on The Cycle Double Cover Conjecture (Conjecture 3.9). In Exercise 9.4.2 it is to be shown that if G is 2-connected with 2-vertex cut S such that each marked S-component of G has a cycle double cover then so has G. Now 2-connected graphs on at most three vertices have cycle double covers (easily verified by checking such graphs), so if The Cycle Double Cover Conjecture is true for all 3-connected graphs (and hence is true for each type of leaf in the decomposition tree), then it is true for all 2-connected graphs. If a graph has connectivity 1 then it has a cut vertex (which is also a separating vertex) and we can consider The Cycle Double Cover Conjecture on the (smaller) components that result from the separation. So the smallest (in terms of the number of vertices) possible counterexample to The Cycle Double Cover Conjecture must be 3-connected (for a counterexample that is 2-connected and not 3-connected has a 3-connected component that is small [in terms of vertices] and also a counterexample). In 1975 and 1976, P. A. Kilpatrick and F. Jeager proved that every 4-edge-connected graph has a cycle double cover (see Bondy and Murty's Theorem 21.24). So if a counterexample exists, then the one with fewest vertices must have connectivity precisely 3.

Note. In Chapter 10, "Planar Graphs," we will classify graphs as planar in terms of the existence of certain subgraphs (see Kuratowski's Theorem, Theorem 10.30). The next result plays a central role. This is due to Carsten Thomassen in 1981; Thomassen is a coauthor, along with Bojan Mohar, of *Graphs on Surfaces*; Baltimore: Johns Hopkins University Press (2001). Notes based on Thomassen and Mohar are available online as supplements to this course.

Theorem 9.10. Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that G/e is 3-connected.

Note. We need a lemma to prove Theorem 9.10.

Lemma 9.11. Let G be a 3-connected graph on at least five vertices, and let $e = xy$ be an edge of G such that G/e is not 3-connected. Then there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G.

Note. We are now ready for the proof of Theorem 9.10.

Note. Theorem 9.10 shows that a 3-connected graph G (on at least five vertices) has an edge e such that the contraction of the edge yields a 3-connected graph G/e. We now introduce a sort of inverse of the contraction which we then show preserves the property of 3-connectedness.

Definition. Let G be a 3-connected graph and let v be a vertex of G of degree at least four. Split v into two vertices, v_1 and v_2, add a new edge e between v_1 and v_2, and distribute the edges of G incident to v among v_1 and v_2 in such a way that v_1 and v_2 each have at least three neighbors in the resulting graph H. The graph H is an *expansion* of G at v.

Note. Since v_1 and v_2 are adjacent to each other, then each picks up at least two edges which were incident to v (which is why v needs to be degree at least four). An example of the expansion of G at v is given in Figure 9.10. If we take such an expansion H, then we have $H/e \cong G$ so that expansion and contraction are, in a sense, inverses of each other. The following is a kind of converse of Theorem 9.10.

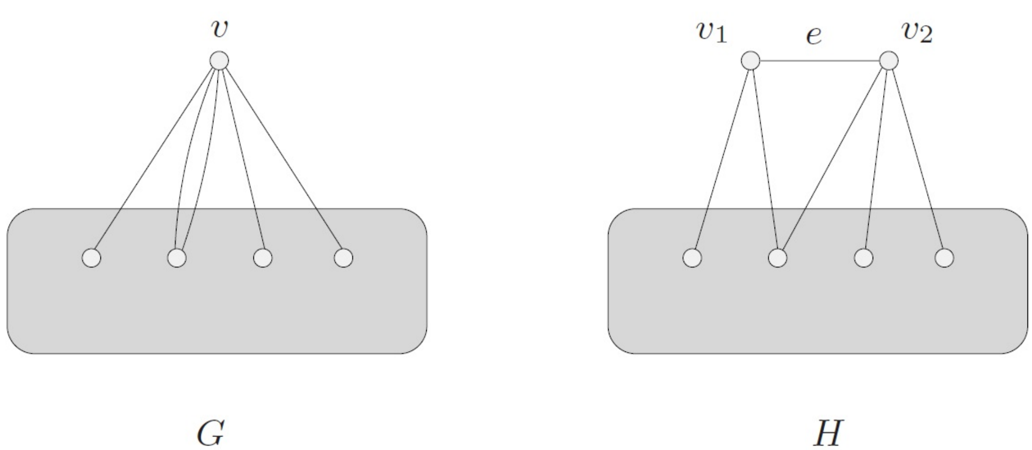

Figure 9.10. An expansion of graph G at vertex v.

Theorem 9.12. Let G be a 3-connected graph, let v be a vertex of G of degree at least four, and let H be an expansion of G at v. Then H is 3-connected.

Note. By Theorem 9.10 and Theorem 9.12, every 3-connected graph G can be obtained by starting with a K_4 and adding edges and vertex expansions. That is, given a 3-connected graph G, there exists a sequence $G_1, G_2, \ldots, G_k$ of graphs such that (i) $G_1 = K_4$, (ii) $G_k = G$, (iii) for $1 \le i \le k-1$, G_{i+1} is obtained by adding an edge to G_i or by expanding G_i at a vertex of degree at least four.

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