Chapter 12. The Cycle Space and Bond Space

of J. A. Bondy and U. S. R. Murty's *Graph Theory with Applications* (1976)

Note. In Section 2.6. Even Subgraphs of J. A. Bondy and U. S. R. Murty's *Graph Theory*, Graduate Texts in Mathematics #244 (Springer, 2008), the *edge space* of a graph G is defined as the vector space over scalar field $GF(2) \cong \mathbb{Z}_2$ with vectors as sets of edges. With $\mathcal{E}(G)$ as the power set of the edge set of G, $\mathcal{E}(G) = \mathcal{P}(E(G))$, we define the the vector sum of $E_1, E_2 \in \mathcal{E}(G)$ as the symmetric difference $E_1 + E_2 = E_1 \triangle E_2$. We define scalar multiplication as $0E_1 = \varnothing$ and $1E_1 = E_1$. In Exercise 2.6.2 of that source, it is to be shown that $\mathcal{E}(G)$ is, in fact, a vector space and it is isomorphic to $(GF(2))^{|}E(G)| = (GF(2))^E$. If $E(G) = \{e_1, e_2, \ldots, e_m\}$ then the *standard basis* for the edge space is $\{e_1\}, \{e_2\}, \ldots, \{e_m\}$. In Theorem 2.6.A it is shown that all even subgraphs of G form a subspace of the edge space, called the *cycle space*. In Theorem 2.6.B it is shown that all edge cuts of G form a subspace of the edge space, called the *bond space*. These names arrive from generating sets of the spaces. In Exercises 2.6.4(a) and 2.6.4(b) it is to be shown that the cycles of G generate the cycle space of G and the bonds of G generate the bond space of G. In Exercise 2.6.4(c) it is to be shown that the bond space of G is the row space of the incidence matrix $\mathbf{M}$ of G over $GF(2)$, and the cycle space of G is the orthogonal complement (in the edge space) of the row space. In these supplemental notes, we cover Section 12.1, Circulations and Potential Differences, from J. A. Bondy and U. S. R. Murty's *Graph Theory with Applications* (Macmillan Press Ltd., 1976). This will give an alternative proof of the graduate text's Exercise 2.6.4(c).

Note. In Section 4.3. Fundamental Cycles and Bonds of Bondy and Murty's graduate text, a "fundamental cycle" and a "fundamental bond" with respect to a given spanning tree of a connected graph are defined. In Exercise 4.3.6(a) it is to be shown that, for a given spanning tree T of a connected graph, the fundamental cycles with respect to T form a basis of the cycle space and the fundamental bonds with respect to T form a basis of the bond space.

Supplement. Section 12.1. Circulations and Potential Differences

Note. In this supplement, we define circulation and potential difference in a digraph D and use these ideas to discuss the cycle space and bond space of D. In the process, we find the dimensions of these spaces.

Definition. Let D be a digraph. A real-valued function f on the arc set $A(D) = A$ is a *circulation in D* if it satisfies the *conservation condition* $f^-(v) = f^+(v)$ for all $v \in V$, where

$$f^+(v) = \sum_{a \in A,\ a \text{ is an arc from } v} f(a) \text{ and } f^-(v) = \sum_{a \in A,\ a \text{ is an arc to } v} f(a).$$

Note. If D represents an electrical network (which we explore in Chapter 20 of Bondy and Murty's graduate text), then circulation f represents the circulation of currents in D. An example of a circulation is given in Figure 12.1.

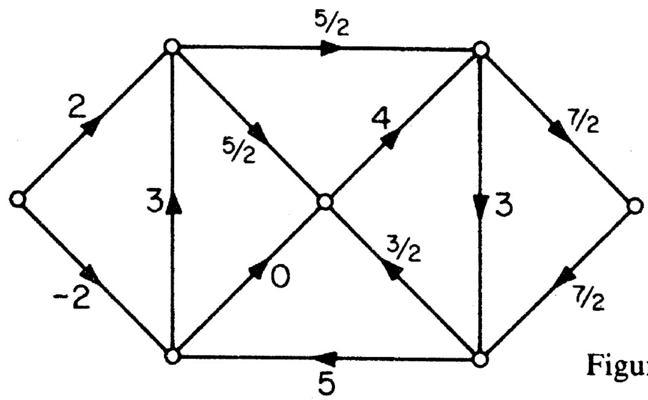


Figure 12.1. A circulation

Note 12.1.A. If f and g are any two circulations on D and $r \in \mathbb{R}$, then $f + g$ and rf are also circulations because for all $v \in V$ we have

$$(f + g)^-(v) = f^-(v) + g^-(v) = f^+(v) + g^+(v) = (f^+ + g^+)(v)$$

and

$$(rf)^-(v) = r(f^-(v)) = r(f^+(v)) = (rf)^+(v).$$

So the conservation condition holds for $f+g$ and rf as well. So the set of circulations on D is closed under (vector) addition and (scalar) multiplication. Therefore the circulations on D form a vector space which we denote as $\mathcal{C}$.

Note 12.1.B. Let C be a cycle in digraph D (that is, the edges a subgraph of D whose underlying undirected graph is a cycle). Consider an assigned orientation of C that makes it a directed cycle and let C^+ denote the set of arcs of C whose direction agrees with this orientation. Define the function f_C on arc set A of D by

$$f_C(a) = \begin{cases} 1 & \text{if} \quad a \in C^+ \\ -1 & \text{if} \quad a \in C \setminus C^+ \\ 0 & \text{if} \quad a \notin C. \end{cases}$$

Then f_C satisfies the conservation condition, because (1) if the two arcs incident with vertex v in cycle C go in the same direction as the directed cycle then $f^-(v) = f^+(v) = 1$, (2) if the two arcs incident with vertex v in cycle C go in the opposite direction as the directed cycle then $f^-(v) = f^+(v) = -1$, and (3) if one arc incident with vertex v in cycle C goes in the same direction as the directed cycle and the other arc incident with vertex v in cycle C goes in the opposite direction as the directed cycle then $f^-(v) = f^+(v) = 0$. Figure 2.1 illustrates this with the cycle given by bold arcs and the assigned orientation given by the curved arc at the upper left:

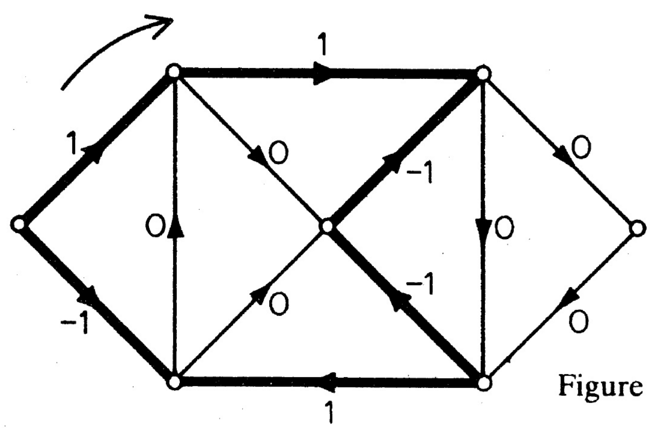


Figure 12.2

Definition. The vector space of Note 12.1.A is the *cycle space* of digraph D.

Note. We'll see later that each circulation on D is a linear combination of the circulations associated with cycles (again, the edges a subgraph of D whose underlying undirected graph is a cycle). This is the reason for the term "cycle space" in the previous definition.

Definition. Given a function p on the vertex set V of digraph D, define the function δp on the arc set A of D such that for arc $a \in A$, where a has tail x and head y, then $\delta p(a) = p(x) - p(y)$. Function g on A is a *potential difference* in D if $g = \delta p$ for some function p on V.

Note. If D is thought of as an electrical network with potential $p(v)$ at v, then δp is the potential difference along the wires of the network (thus the terminology "potential difference"). Figure 12.3 gives an example of a function p on the vertices of D and the corresponding potential difference δp on the arcs of D.

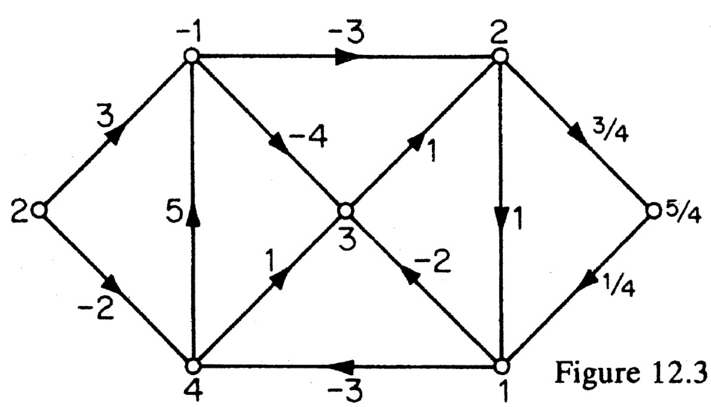


Figure 12.3. A potential difference

Note 12.1.C. Let $\mathcal{B}$ denote the set of all potential differences on digraph D. Notice that a sum and a scalar multiple of a potential difference is again a potential difference. This is because

$$g_1(a) + g_2(a) = \delta p_1(a) + \delta p_2(a) = (p_1(x) - p_1(y)) + (p_2(x) - p_2(y))$$

$$= (p_1(x) + p_2(x)) - (p_1(y) + p_2(y)) = \delta(p_1 + p_2)(a)$$

and

$$rg(a) = r\delta p(a) = r(p(x) - p(y)) = (rp(x) - rp(y)) = \delta(rp)(a).$$

So the set of potential differences on D is closed under (vector) addition and (scalar) multiplication. Therefore the potential differences on D form a vector space which we denote as $\mathcal{B}$.

Note 12.1.D. For S a nonempty subset of the vertex set of digraph D, denote $\overline{S} = V \setminus S$. Let $B = D[S, \overline{S}] = [S, \overline{S}]$ be a bond of D (that is, a minimal edge cut). Define g_B by

$$g_B(a) = \begin{cases} 1 & \text{if} \quad a \in (S, \overline{S}) \\ -1 & \text{if} \quad a \in (\overline{S}, S) \\ 0 & \text{if } a \notin B, \end{cases}$$

where $(S, \overline{S})$ is the set of arcs in D with tail in S and head in $\overline{S}$ (and similarly for $(\overline{S}, S)$). Then $g_B = \delta p$ where $p(v) = \begin{cases} 1 & \text{if} \quad v \in S \\ 0 & \text{if} \quad v \in \overline{S}. \end{cases}$ Figure 12.4 below gives a potential difference associated with the bond given by bold arcs. The set $\overline{S}$ consists of the two vertices in the upper right of the digraph.

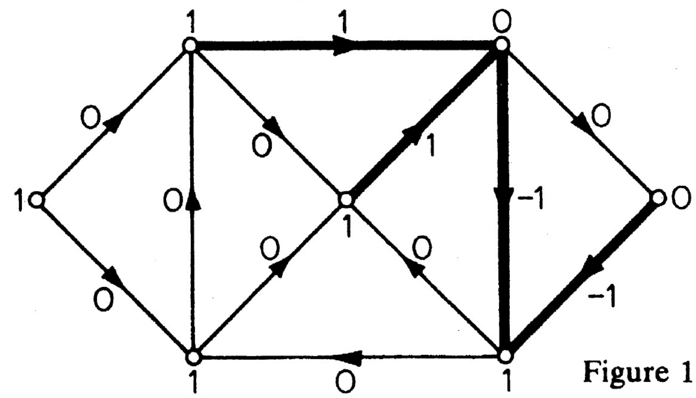


Figure 12.4

Definition. The vector space of Note 12.1.C is the *bond space* of digraph D.

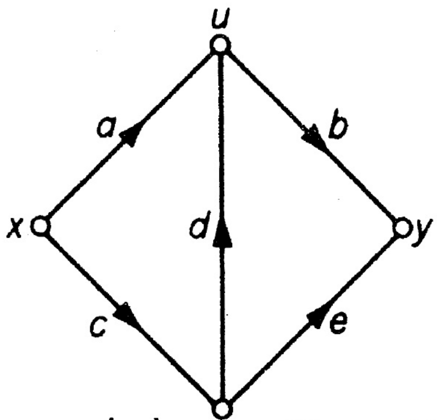
Note. We'll see later that each potential difference on D is a linear combination of potential differences associated with bonds. This is the reason for the term "bond space" in the previous definition.

Definition. With each vertex v of digraph D, define the function m_v on the arc set A of D by

$$m_v(a) = \begin{cases} 1 & \text{if } a \text{ is a link and } v \text{ is the tail of } a \\ -1 & \text{if } a \text{ is a link and } v \text{ is the head of } a \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that a link is a nonloop.) The *incidence matrix* of D is the matrix $\mathbf{M}$ whose rows are indexed by the vertices of D and whose columns are indexed by the arcs of D, and the (v,a) entry of $\mathbf{M}$ is $m_v(a)$.

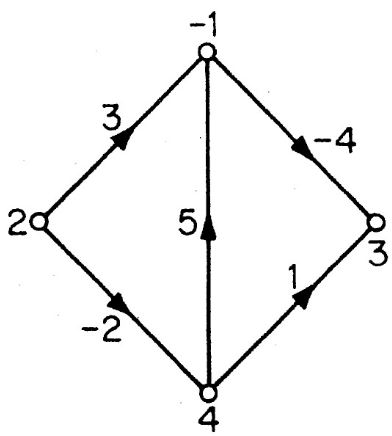
Note. Figure 12.5 gives an example of a small digraph and its incidence matrix.

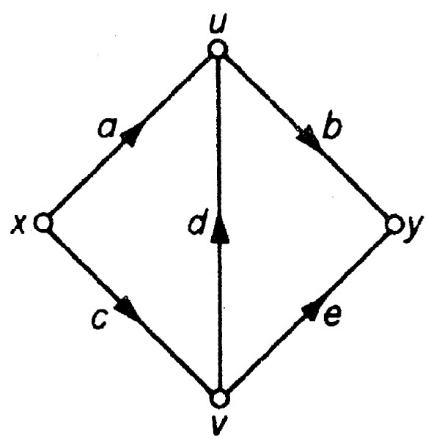


$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline x & 1 & 0 & 1 & 0 & 0 \\ u & -1 & 1 & 0 & -1 & 0 \\ v & 0 & 0 & -1 & 1 & 1 \\ y & 0 & -1 & 0 & 0 & -1 \end{array} \quad \text{(b)}$$

(a) **Figure 12.5.** **(*a*) *D*; (*b*) the incidence matrix of *D***

Note. Next, we will relate the incidence matrix of digraph D to the row space and its orthogonal complement in the edge space. First, we illustrate the fact that a linear combination of the rows of the incidence matrix $\mathbf{M}$ is a potential difference. Consider the following figure, which is parts of Figures 12.3 and 12.5.





$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline x & 1 & 0 & 1 & 0 & 0 \\ u & -1 & 1 & 0 & -1 & 0 \\ v & 0 & 0 & -1 & 1 & 1 \\ y & 0 & -1 & 0 & 0 & -1 \end{array}$$

Here, the rows of $\mathbf{M}$ are $[1, 0, 1, 0, 0]$, $[-1, 1, 0, -1, 0]$, $[0, 0, -1, 1, 1]$, $[0, -1, 0, 0, -1]$.

Consider the linear combination

$$2[1,0,1,0,0]-[-1,1,0,-1,0]+4[0,0,-1,1,1]+3[0,-1,0,0,-1]=[3,-4,-2,5,1].$$

This gives the potential difference g on the arcs of $g(a)=3$, $g(b)=-4$, $g(c)=-2$, $g(d)=5$, $g(e)-1$ (that is, $[g(a),g(b),g(c),g(d),g(e)]=[3,-4,-2,5,1]$), as shown in the figure (left). Notice that the coefficients in the linear combination are just the values of p on the vertices: $p(x)=2$, $p(u)=-1$, $p(v)=4$, and $p(y)=3$, as in the figure (left).

Theorem 12.1. Let $\mathbf{M}$ be the incidence matrix of a digraph D. Then $\mathcal{B}$ is the row space of $\mathbf{M}$ and $\mathcal{C}$ is its orthogonal complement.

Note. As seen in the proof of Theorem 12.1, the bond space $\mathcal{B}$ is the row space of incidence matrix $\mathbf{M}$ and the cycle space $\mathcal{C}$ is the nullspace of $\mathbf{M}$. By the rank equation, we have that the dimension $\mathcal{B}$ plus the dimension of $\mathcal{C}$ equals the number of columns of $\mathbf{M}$ (namely, it equals the number of arcs in D). For more on the vector spaces associated with a matrix, see my online notes for Linear Algebra (MATH 2010) on Section 2.2. The Rank of a Matrix.

Definition. The *support* of a function f on A is the set of elements of A at which the value of f is nonzero. We denote the support of f by $\|f\|$.

Note. You also encounter the support of a function in Lebesgue integration theory. See my online notes for Real Analysis 1 on Section 4.3. The Lebesgue Integral of a Measurable Nonnegative Function. Here, functions of finite support (i.e., those that nonzero on a set of finite measure) are used in connection with sets defined on a set of finite measure. In our setting, we consider functions defined finite sets, so measure is not of concern here. However, we can relate the support of circulations and potential differences to cycles and bonds, respectively, as follows.

Lemma 12.2.1. If f is a nonzero circulation (that is, f is not identically zero), then $\|f\|$ contains a cycle.

Lemma 12.2.2. If g is a nonzero potential difference (that is, g is not identically zero), then $\|g\|$ contains a bond.

Note. In Theorem 12.1 we have a spanning set for the bond space (it is the rows of the incidence matrix $\mathbf{M}$) and since the cycle space is the nullspace of $\mathbf{M}$ then we can find a spanning set for it as well. We are interested in bases for these spaces, so we introduce the following.

Definition. A matrix $\mathbf{B}$ is a *basis matrix* of $\mathcal{B}$ if the rows of $\mathbf{B}$ form a basis for $\mathcal{B}$; a basis matrix of $\mathcal{C}$ is similarly defined.

Note. If $\mathbf{R}$ is a matrix whose columns are labeled with the elements of set A and if $S \subseteq A$, then we denote the submatrix of $\mathbf{R}$ consisting only of those columns of $\mathbf{R}$ labeled with elements in S as $\mathbf{R}|S$. We read $\mathbf{R}|S$ as "matrix $\mathbf{R}$ restricted to columns in S." In the next result, we take a step in the direction of linear independence (and hence in the direction of finding bases for $\mathcal{B}$ and $\mathcal{C}$. We start with the incidence matrix $\mathbf{M}$ of digraph D which has rows indexed by the vertices of D and columns indexed by the arcs of D.

Theorem 12.2. Let $\mathbf{B}$ and $\mathbf{C}$ be basis matrices of $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for any $S \subseteq A$:

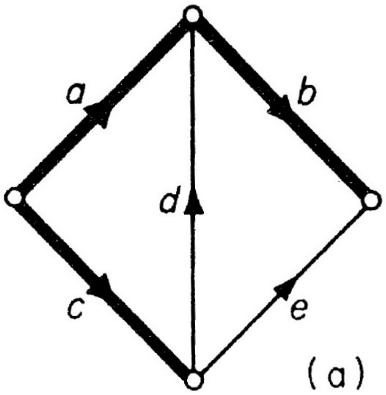
(i) the columns of $\mathbf{B}|S$ are linearly independent if and only if S is acyclic, and

(ii) the columns of $\mathbf{C}|S$ are linearly independent if and only if S contains no bond.

Note. Now that we can classify linearly independent columns of $\mathbf{B}$ and $\mathbf{C}$, we can find maximal linearly independent sets of columns of these matrices, and hence we can find the dimensions of $\mathcal{B}$ and $\mathcal{C}$.

Corollary 12.2. Let D be a digraph. The dimensions of the bond space $\mathcal{B}$ and the cycle space $\mathcal{C}$ are given by $\dim(\mathcal{B}) = \nu - \omega$ and $\dim(\mathcal{C}) = \varepsilon - \nu + \omega$, where ν is the number of vertices of D, ε is the number of arcs of D, and ω is the number of connected components of D.

Note 12.1.E. Let T be a maximal forest of digraph of D. If a is an arc of the complement of T, $\overline{T}$, then (because T is maximal) $T + a$ contains a unique cycle (that is, a digraph whose underlying graph is a cycle). In the terminology of Bondy and Murty's graduate level text, $\overline{T}$ would be called the *coforest* of T (see Section 4.3. Fundamental Cycles and Bonds of that reference). Let C_a denote this cycle and let $f_a = f_{C_a}$ denote the circulation corresponding to C_a, as given in Note 12.1.B, defined so that $f_a(a) = f_{C_a}(a) = 1$ (that is, choose the orientation of C_a as described in Note 12.1.B such that this $f_a(a) = 1$). Let $\mathbf{C}$ be the $(\varepsilon - \nu + \omega) \times \varepsilon$ whose rows are f_a where $a \in \overline{T}$ (so the rows are of the form $[f_a(a_1), f_a(a_2), \ldots, f_a(a_\varepsilon)]$). Now $\mathbf{C}|\overline{T}$ is an identity matrix; notice $f_a(a) = 1$ and $f_a(b) = 0$ for $b \in \overline{T}$ and $b \neq a$. Since T is a maximal forest, then by Exercise 2.2.4 of Bondy and Murty's *Graph Theory with Applications* or by Exercise 4.1.1 of Bondy and Murty's graduate level *Graph Theory*, T has $\nu - \omega$ arcs. Hence $\overline{T}$ contains $\varepsilon - (\nu - \omega) = \varepsilon - \nu + \omega$ arcs and hence $\text{rank}(\mathbf{C}) = \varepsilon - \nu + \omega$. Also, each row of $\mathbf{C}$ is a circulation by the choice of $f_a = f_{C_a}$. So the $\varepsilon - \nu + \omega$ rows of $\mathbf{C}$ are linearly independent and, since each is a circulation, the row space of $\mathbf{C}$ is the cycle space $\mathcal{C}$; that is, $\mathbf{C}$ so constructed is a basis matrix for $\mathcal{C}$. This is similar to the approach in the graduate level text's Section 4.3. Fundamental Cycles and Bonds (though undirected connected graphs are considered there); the cycles C_a here correspond to the "fundamental cycles with respect to T" there. Figure 12.6(a) below gives a tree (with bold arcs) in a digraph, and Figure 12.6(b) gives the basis matrix of $\mathcal{C}$ corresponding to the tree. Notice that $\mathbf{C}|\overline{T} = \mathbf{C}|\{d, e\}$ is a 2×2 identity matrix.



(a)

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline f_d & -1 & 0 & 1 & 1 & 0 \\ f_e & -1 & -1 & 1 & 0 & 1 \end{array}$$

(b)

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline g_a & 1 & 0 & 0 & 1 & 1 \\ g_b & 0 & 1 & 0 & 0 & 1 \\ g_c & 0 & 0 & 1 & -1 & -1 \end{array}$$

(c)

Figure 12.6

Note 12.1.F. We now turn our attention to the bond space. If a is an arc of T, then $\overline{T}+a$ contains a unique bond (by Theorem 2.6(ii) of Bondy and Murty's *Graph Theory with Applications* or by Lemma 4.3.A and Note 4.3.B of Bondy and Murty's graduate level *Graph Theory* in Section 4.3. Fundamental Cycles and Bonds). Let B_a denote this bond and let $g_a = g_B$ be the potential difference corresponding to B_a, as given in Note 12.1.D, defined so that $g_a(a) = 1$ (this is called the *fundamental bond* with respect to T and a in the graduate level text in Section 4.3. Fundamental Cycles and Bonds, where graphs are considered, as opposed to digraphs). Now $\mathbf{B}|T$ is an identity matrix; notice by Note 12.1.D that $g_B(a) = 1$ and $g_B(a) = 0$ if $a \notin B$. The $(\nu - \omega) \times \varepsilon$ matrix $\mathbf{B}$ whose rows are the g_a for $a \in T$ is, analogous to the cycle space case above, a basis matrix of the bond space $\mathcal{B}$ called the basis matrix *corresponding to* T. Figure 12.6(c) gives an example for the tree of Figure 12.6(a). Notice that $\mathbf{B}|T = \mathbf{B}|\{a, b, c\}$ is a 3×3 identity matrix.

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