Chapter 12. The Cycle Space and Bond Space

of J. A. Bondy and U. S. R. Murty's *Graph Theory with Applications* (1976)

Note. In Section 4.2. Spanning Trees of J. A. Bondy and U. S. R. Murty's *Graph Theory*, Graduate Texts in Mathematics #244 (Springer, 2008), we see that the number of labeled trees on n vertices (and hence the number of spanning trees of K_n) is n^{n-2} by Cayley's Formula (Theorem 4.8). In these supplemental notes, we cover Section 12.2, The Number of Spanning Trees, from J. A. Bondy and U. S. R. Murty's *Graph Theory with Applications* (Macmillan Press Ltd., 1976) and give a formula for the number of spanning trees in an arbitrary connected graph.

Supplement. Section 12.2. The Number of Spanning Trees

Note. In this supplement, we determine the number of spanning trees in a graph in terms of the basis matrix $\mathbf{B}$ for the bond space $\mathcal{B}$.

Note 12.2.A. Let T be a spanning tree of connected graph G. We put an arbitrary orientation on G ro produce digraph D (so that we can use the ideas from Supplement. Section 12.1, "Circulations and Potential Differences" on the bond space. Let $\mathbf{B}$ be the basis matrix of $\mathcal{B}$ corresponding to T. By Theorem 12.2 of the supplement, if S is a subset of arc set A with $|S| = v - 1$ then the square submatrix $\mathbf{B}|S$ is nonsingular if and only if S is a spanning tree of G (the linear independence of the columns of $\mathbf{B}|S$ is given by Theorem 12.2(i) and by Corollary 12.2 the dimension of $\mathcal{B}$ is $v-1$ since G is connected). So there is a one-to-one correspondence between spanning trees of G and the nonsingular square submatrices of $\mathbf{B}$ of size $(v-1)\times(v-1)$. That is, the number of spanning trees of G is equal to the number of nonsingular submatrices of $\mathbf{B}$ of size $(v-1)\times(v-1)$.

Definition. A matrix is *unimodular* if all of its full square submatrices have determinants 0, $+1$, or -1.

Theorem 12.3. Let G be a graph and T a spanning tree of G. Let D be any orientation of G and let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T. Then $\mathbf{B}$ is a unimodular matrix.

Note 12.2.B. For the next proof, we need a result concerning the determinant of the matrix product $\mathbf{AB}$, where $\mathbf{A}$ is $m\times n$ and $\mathbf{B}$ is $n\times m$ (where $m\le n$). Notice that $\mathbf{A}$ is $m\times m$. In G. Hadley's *Linear Algebra*, Addison Wesley (1961), the following is proved (see Section 3-17, "Determinant of the Product of Rectangular Matrices," and page 102).

Hadley's Theorem. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix (where $m \leq n$), then $\det(\mathbf{AB})$ can be represented as the sum of $\binom{n}{m}$ terms. Each term is the product of two determinants of $m \times m$ submatrices of $\mathbf{A}$ and $\mathbf{B}$, respectively. A given determinant of an $m \times m$ submatrix of $\mathbf{A}$ is formed from the columns $j_1, j_2, \ldots, j_m$ of $\mathbf{A}$, and the corresponding determinant of an $m \times m$ submatrix of $\mathbf{B}$ is formed from the rows $j_1, j_2, \ldots, j_m$ of $\mathbf{B}$.

We can apply Hadley's Theorem to the product $\mathbf{BB}'$, since $\mathbf{B}$ a basis matrix for $\mathcal{B}$ and $\dim(\mathcal{B}) = \nu - 1$; the rows of $\mathcal{B}$ form a basis for $\mathcal{B}$ so that $\mathbf{B}$ is $m \times n = (\nu - 1) \times \varepsilon$. Since G is connected, then it contains a spanning tree on ν vertices with $\nu - 1$ arcs (by Theorem 4.3 of Bondy and Murty's graduate text in Section 4.1. Forests and Trees), so that in G we have $m = \nu - 1 \leq \varepsilon = n$. Hadley's Theorem implies that $\det(\mathbf{BB}')$ is a sum of $\binom{n}{m} = \binom{\varepsilon}{\nu-1}$ terms, each of which is the product of a determinant of a $(\nu - 1) \times (\nu - 1)$ submatrix of $\mathbf{B}$ based on $\nu - 1$ of the columns of $\mathbf{B}$, and the corresponding determinant of a $(\nu - 1) \times (\nu - 1)$ submatrix of $\mathbf{B}'$ based on the corresponding rows of $\mathbf{B}'$. But $\mathbf{B}'$ is the transpose of $\mathbf{B}$, so the two submatrices have the same determinant. We choose the submatrices by choosing $\nu - 1$ columns of $\mathbf{B}$, or equivalently by choosing a subset S of the arc set A where $|S| = \nu - 1$. Then the $(\nu - 1) \times (\nu - 1)$ submatrix of $\mathbf{B}$ is $\mathbf{B}|S$. Then Hadley's Theorem Letting S range over all subsets of A of cardinality $\nu - 1$, Hadley's Theorem gives

$$\det(\mathbf{BB}') = \sum_{S \subseteq A,\, |S| = \nu - 1} (\det(\mathbf{B}|S))^2. \tag{12.17}$$

Theorem 12.4. Let G be a graph and T a spanning tree of G. Let D be any orientation of G and let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T. The number of spanning trees of G is $\tau(G) = \det(\mathbf{B}\mathbf{B}')$, where $\mathbf{B}'$ is the transpose of $\mathbf{B}$.

Note 12.2.C. Similar to Note 12.2.A, Theorem 12.3, and Theorem 12.4, one can show that if $\mathcal{C}$ is a basis matrix for the cycle space $\mathcal{C}$ corresponding to a tree, then $\mathbf{C}$ is unimodular (similar to Theorem 12.3) and the number of spanning trees of G is $\tau(G) = \det(\mathbf{C}\mathbf{C}')$ (similar to Theorem 12.4).

Note 12.2.C. For the final result in this section, we need to review some terminology and results from Theory of Matrices (MATH 5090). In Section 3.1. Basic Definitions and Notations, an $n \times m$ matrix $\mathbf{A}$ is *partitioned* into four matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{22}$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$ have the same number of rows (say r_1); $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ have the same number of rows (say r_2); $\mathbf{A}_{11}$ and $\mathbf{A}_{21}$ have the same number of columns (say c_1); and $\mathbf{A}_{12}$ and $\mathbf{A}_{22}$ have the same number of columns (say c_2). In Theorem 3.1.G of the Theory of Matrices notes, it is shown that the determinant of partitioned matrix with an off-diagonal zero matrix can be computed as =

$$\det \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{V} & \mathbf{W} \end{bmatrix} = \det \begin{bmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} = \det(\mathbf{T})\det(\mathbf{W}),$$

where $\mathbf{T}$ and $\mathbf{W}$ are square. Products of partitioned matrices are considered in

Theory of Matrices in Theorem 3.2.2 of Section 3.2. Multiplication of Matrices and Multiplication of Vectors and Matrices, where it is shown for appropriately sized matrices:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{AE}+\mathbf{BG} & \mathbf{AF}+\mathbf{BH} \\ \mathbf{CE}+\mathbf{DG} & \mathbf{CF}+\mathbf{DH} \end{bmatrix}.$$

Corollary 12.4. Let G be a graph and T a spanning tree of G. Let D be any orientation of G, let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T, and let $\mathbf{C}$ be the basis matrix of the cycle space $\mathcal{C}$ corresponding to T. The number of spanning trees of G is $\tau(G) = \pm\det \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}$.

Note 12.2.D. Since Theorem 12.2 holds for all basis matrices of the bond space $\mathcal{B}$, so Theorem 12.4 holds for any basis matrix of $\mathcal{B}$, $\mathbf{B}$, that is unimodular. It is to be shown in Exercise 12.2.1(a) of J. A. Bondy and U. S. R. Murty's *Graph Theory with Applications* that a matrix $\mathbf{K}$ obtained by deleting any one row of the incidence matrix $\mathbf{M}$ of digraph D is unimodular. From this we have that the number of spanning trees of G can also be expressed as $\tau(G) = \det(\mathbf{KK}')$. This expression for $\tau(G)$ is known as the *Matrix-Tree Theorem* and originally due to Gustav Kirchhoff (March 12, 1824–October 17, 1887); it appeared in "Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird [On the Resolution of the Equations to which One is Led when Investigating the Linear Distribution of Galvanic Currents]" *Annalen der Physik und Chemie* **72**(12), 497–508 (1847); the front page of the paper can be viewed on the Wiley Online Library webpage. The Matrix-Tree Theorem is covered in J. A. Bondy and U. S. R. Murty's graduate level *Graph Theory* in Chapter 20, "Electrical Networks," and Section 20.4, "The Matrix-Tree Theorem."

Revised: 12/24/2022