Supplement. Graph Decompositions: Triple Systems

Note. In this supplement, we consider a simple graph G (often K_n) and various isomorphic decompositions into copies of a given small graph. We start with Steiner triple systems.

Definition. A *Steiner triple system* of order n is an isomorphic decomposition of $G = K_n$ into a family $\mathcal{F}$ of subgraphs of G such that each $F \in \mathcal{F}$ is isomorphic to a 3-cycle. The members of $\mathcal{F}$ are called the *blocks* of the Steiner triple system. We denote a Steiner triple system of order n as a $STS(n)$.

Note. The definition of a Steiner triple system in the realm of design theory is slightly different (though equivalent) to the above. Quoting from Lindner and Rodger's *Design Theory* (see page 1) [8]:

> A *Steiner triple system* is an ordered pair (S, T) where S is a finite set of *points* or *symbols*, and T is a set of 3-element subsets of S called *triples*, such that each pair of distinct elements of S occurs together in exactly one triple of T. The *order* of a Steiner triple system (S, T) is the size of the set S, denoted $|S|$.

I have online notes for an undergraduate/graduate level class on Design Theory.

Note. Lindner and Rodger [8, page 1] say that Steiner triple systems were apparently first defined by W. S. B. Wool-House in 1844 in the *Lady's and Gentlemen's Diary* as "Prize Question 1733." The problem was ultimately solved by Thomas P.

Kirkman (1806–1895) in "On a Problem of Combinations," *Cambridge and Dublin Mathematics Journal*, **2** (1847), 191–204. Ironically, Steiner triple systems are named for Jakob Steiner (1796–1863), a Swiss mathematician working in Berlin most of his career, who gave necessary conditions for their existence and published it in "Combinatorische Aufgabe," *Journal für die Reine und angewandte Mathematik* (*Crelles Journal*), **45** (1853), 181–182. The strange dates on the necessary conditions of Steiner and the sufficiency of Kirkman are explained by a lack of communication between mainland Europe and the British Isles at the time—this likely results from fallout from the argument between Newton and Leibniz over who deserves the credit for inventing/discovering calculus.

Jakob Steiner (1796–1863)

Thomas P. Kirkman (1806–1895)

These images are from the MacTutor History of Mathematics Archive (accessed 3/6/2020).

Theorem. (Kirkman, 1847) A Steiner triple system of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.

Examples. Let the vertex set of K_7 be $\{0,1,2,3,4,5,6\}$. We represent the 3-cycle on vertices a,b,c with edges ab, bc, and ac as $[a,b,c] = [b,c,a] = [c,a,b] = [c,b,a] = [a,c,b] = [b,a,c]$. Then the blocks of a Steiner triple system of order 7 is given by:

$$\mathcal{F} = \{[0,1,3],[1,2,4],[2,3,5],[3,4,6],[4,5,0],[5,6,1],[6,0,2]\}.$$

With the vertex set of K_9 as $\{0,1,2,\ldots,7,8\}$, the blocks of a Steiner triple system of order 9 is given by:

$$\mathcal{F} = \{[0,1,2],[0,3,6],[0,4,8],[0,5,7],[3,4,5],[1,4,7],$$

$$[1,5,6],[1,3,8],[6,7,8],[2,5,8],[2,3,7],[2,4,6]\}.$$

There is a clear pattern in the blocks of the $STS(7)$, but the pattern for the $STS(9)$ is not clear.

Note. Let's establish the necessary conditions first given by Steiner. The argument is based on the number of edges and the degrees of vertices.

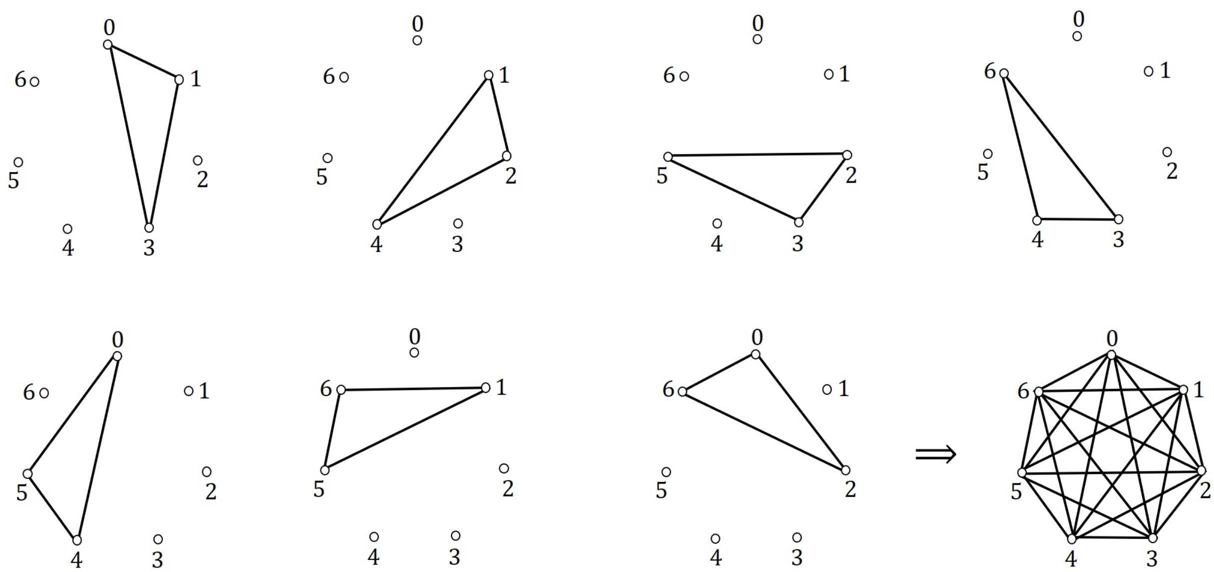
Lemma T.1. If $STS(n)$ exists then $n \equiv 1$ or $3 \pmod 6$.

Note. We now explore constructions that show the necessary conditions of Lemma T.1 are, in fact, sufficient.

Definition. Let K_n be a complete graph with vertex set $V(K_n) = \{0,1,2,\ldots,n-1\}$. With edge $xy \in E(K_n)$, associate the *difference*:

$$|x-y|_n = \min\{(x-y)(\text{mod } n),(y-x)(\text{mod } n)\}.$$

Note. The set of differences associated with the edges of K_n is $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Under the permutation $\alpha = (0, 1, \ldots, n-1)$ (a "cyclic permutation") we have that if $e = ab$ is any edge with associated difference $d = |a - b|_n$, then every edge in K_n with associated difference d is in the orbit of edge e, where the orbit of edge e is $\{\alpha^i(a)\alpha^i(b) \mid i = 0, 1, \ldots, n\} \subseteq E(K_n)$. This allows us to address certain decomposition problems in terms of partitions of the set of differences. For example, the set of differences associated with K_7 is $\{1, 2, 3\}$ and we can consider the 3-cycle $[0, 1, 3]$. The differences associated with the edges of this cycle are $|1 - 0|_7 = 1$, $|3 - 1|_7 = 2$, and $|3 - 0|_7 = 3$. So each difference associated with edges in K_7, appears exactly once. Therefore, there is an isomorphic decomposition of K_7 into 3-cycles given by $\{[0, 1, 3], [1, 2, 4], \ldots, [6, 0, 2]\}$. This gives an easy justification of the claimed decomposition of a $STS(7)$ given above. Geometrically, we have:



Note. For a $STS(13)$, we observe that the set of differences associated with edges of K_{13} is $\{1, 2, \ldots, 6\}$. Notice that the two 3-cycles $[0, 1, 4]$ and $[0, 2, 7]$ have edges with associated differences $|1-0|_{13} = 1$, $|4-1|_{13} = 3$, $|4-0|_{13} = 4$ and $|2-0|_{13} = 2$, $|7-2|_{13} = 5$, $|7-0|_{13} = 6$, respectively. So the orbits of the 3-cycles $[0, 1, 4]$ and $[0, 2, 7]$ under the permutation $\alpha = (0, 1, \ldots, 12)$ generate a $STS(13)$:

$$\{[i \pmod{13}, (1+i) \pmod{13}, (4+i) \pmod{13}],$$

$$[i \pmod{13}, (2+i) \pmod{13}, (7+i) \pmod{13}] \mid i = 0, 1, \ldots, 12\}.$$

In the remainder of this section, we assume such vertex labels are reduced modulo n so that we can express the decomposition as

$$\{[i, 1+i, 4+i], [i, 2+i, 7+i] \mid i = 0, 1, \ldots, 12\}.$$

Notice that the differences $d_1 = 1$, $d_2 = 3$, $d_3 = 4$ associated with the edges of $[0, 1, 4]$ satisfy $d_1 + d_2 = d_3$; the differences $d_1 = 2$, $d_2 = 5$, $d_3 = 6$ associated with the edges of $[0, 2, 7]$ satisfy $d_1 + d_2 + d_3 \equiv 0 \pmod{13}$.

Note T.A. If 3-cycle $[a, b, c]$ in K_n has edges with associated distinct differences d_1, d_2, d_3 then either $d_1 + d_2 = d_3$ or $d_1 + d_2 + d_3 \equiv 0 \pmod{n}$. This is required because the third edge of the 3-cycle must have the "first" vertex as one of its ends. So often times finding a 3-cycle decomposition of K_n (and hence a $STS(n)$) is equivalent to partitioning the set of differences $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ into (difference) triples d_1, d_2, d_3 such that either $d_1 + d_2 = d_3$ or $d_1 + d_2 + d_3 \equiv 0 \pmod{n}$.

Note. A $STS(15)$ exists. The set of differences associated with K_{15} is $\{1, 2, \ldots, 7\}$ (not a multiple of 3). But since $3 \mid 15$, there is a trick! We take the 3-cycle $[0, 5, 10]$ and notice that each edge has associated difference 5. Under powers of the permutation $\alpha = (0, 1, \ldots, 14)$ we get the 3-cycles $[0, 5, 10]$, $[1, 6, 11]$, $[2, 7, 12]$, $[3, 8, 13]$, and $[4, 9, 14]$ (notice that applying α to the last 3-cycle yields $[5, 10, 0] = [0, 5, 10]$). This 3-cycle is said to have a "short orbit." This leaves the differences $1, 2, 3, 4, 6, 7$ unaddressed. The 3-cycle $[0, 1, 4]$ has associated differences 1, 3, 4 (and $1 + 3 = 4$), and the 3-cycle $[0, 2, 8]$ has associated differences 2, 6, 7 (and $2 + 6 + 7 = 15 \equiv 0 \pmod{15}$).

Note. The previous two notes give the conditions on the differences associated with a 3-cycle in K_n. If d_1, d_2, d_3 are distinct and associated with a 3-cycle, then either $d_1 + d_2 = d_3$ or $d_1 + d_2 + d_3 \equiv 0 \pmod{n}$. If $n \equiv 0 \pmod{3}$, then the single difference $d = n/3$ can be associated with short orbit 3-cycle $[0, n/3, 2n/3]$. So if the difference set $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ can be partitioned into triples d_1, d_2, d_3 where either $d_1 + d_2 = d_3$ or $d_1 + d_2 + d_3 \equiv 0 \pmod{n}$, AND when $n \equiv 0 \pmod{3}$ the difference $d = n/3$ can be part of the partition, then there is a $STS(n)$. A Steiner triple system constructed in this way admits a permutation $\alpha = (0, 1, \ldots, n-1)$ as an "automorphism" and the Steiner triple system is called *cyclic*. A collection of triples which contains exactly once every difference associated with the edges of K_n is called a collection of *base blocks* for the cyclic $STS(n)$. In fact, for all $n \equiv 1$ or 3 (mod 6) (this is the necessary condition for the existence of a $STS(n)$ by Lemma T.1) there exists a cyclic $STS(n)$ *except* for $n = 9$. This technique is broken into smaller parts in the following discussion.

Note. By Lemma T.1, a necessary condition for the existence of a $STS(n)$ is $n \equiv 1$ or 3 (mod 6). In 1897, Lothar Heffter [4] stated two difference problems:

Heffter's First Difference Problem. For $n \equiv 1 \pmod 6$, say $n = 6k+1$, partition the set $\{1, 2, \ldots, 3k\}$ into triples such that in each triple either the sum of two numbers equals the third or the sum of the three equals n.

Heffter's Second Difference Problem. For $n \equiv 3 \pmod 6$, say $n = 6k+3$, partition the set $\{1, 2, \ldots, 2k, 2k+2, 2k+3, \ldots, 3k+1\}$ into triples such that in each triple either the sum of two numbers equals the third or the sum of the three equals n.

A solution of Heffter's First Difference Problem is equivalent to the existence of a cyclic $STS(6k+1)$, and a solution of Heffter's Second Difference Problem is equivalent to the existence of a cyclic $STS(6k+3)$. These problems were first solved in 1939 by Rose Peltesohn [12]. We present a solution to Heffter's First Difference Problem here based on Th. Skolem's ideas of (A,k)-systems and (B,k)-systems [14].

Definition. An (A,k)-system (where $k \in \mathbb{N}$) is a partition of the set $\{1, 2, \ldots, 2k\}$ into distinct pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, 2, \ldots, k$. A (B,k)-system (where $k \in \mathbb{N}$) is a partition of the set $\{1, 2, \ldots, 2k-1, 2k+1\}$ into distinct pairs (a_r, b_r) with $b_r = a_r + r$ for $r = 1, 2, \ldots, k$.

Note. If an (A, k)-system exists, then the triples $r, a_r + k, b_r + k$ for $r = 1, 2, \ldots, k$ is a solution to Heffter's First Difference Problem. This is because $b_r = a_r + r$ implies $b_r + k = (a_r + k) + r$ and, since the pairs (a_r, b_r) for $r = 1, 2, \ldots, k$ partition $\{1, 2, \ldots, 2k\}$, then the triples $r, a_r + k, b_r + k$ for $r = 1, 2, \ldots, k$ partition the set $\{1, 2, \ldots, k, k + 1, k + 2, \ldots, 3k\}$. So the existence of an (A, k)-system implies the existence of a cyclic $STS(6k + 1)$.

Note. If a (B, k)-system exists, then the triples $r, a_r + k, b_r + k$ for $r = 1, 2, \ldots, k$ is a solution to Heffter's First Difference Problem. This is because $b_r = a_r + r$ implies $b_r + k = (a_r + k) + r$ and, since the pairs (a_r, b_r) for $r = 1, 2, \ldots, k$ partition $\{1, 2, \ldots, 2k-1, 2k+1\}$, then the triples r, a_r+k, b_r+k for $r = 1, 2, \ldots, k$ partition the set $\{1, 2, \ldots, k, k+1, k+2, \ldots, 3k-1, 3k+1\}$. Notice that the value $3k+1$ as a difference is the same as the difference $3k$ in K_{6k+1} (since $|3k + 1|_{6k+1} = 3k$). So the existence of a (B, k)-system implies the existence of a cyclic $STS(6k + 1)$.

Note. Th. Skolem [14] gave necessary and sufficient conditions for the existence of an (A, k)-system, as follows.

Lemma T.2. An (A, k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$.

Note. Th. Skolem [14] conjectured the following necessary and sufficient conditions for the existence of a (B, k)-system in 1957; this was proved by Edward S. O'Keefe [11] in 1961.

Lemma T.3. A (B,k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$.

Lemma T.4. There exists a cyclic $STS(n)$ for all $n \equiv 1 \pmod 6$.

Definition. A (C,k)-system (where $k \in \mathbb{N}$) is a partition of the set $\{1,2,\ldots,k,k+2,k+3,\ldots 2k+1\}$ into distinct pairs (a_r,b_r) such that $b_r = a_r + r$ for $r = 1,2,\ldots,k$. A (D,k)-system (where $k \in \mathbb{N}$) is a partition of the set $\{1,2,\ldots,k,k+2,k+3,\ldots,2k,2k+2\}$ into distinct pairs (a_r,b_r) with $b_r = a_r + r$ for $r = 1,2,\ldots,k$.

Note. If a (C,k)-system exists, then the triples $r, a_r + k, b_r + k$ for $r = 1,2,\ldots,k$ is a solution to Heffter's Second Difference Problem. This is because $b_r = a_r + r$ implies $b_r + k = (a_r + k) + r$ and, since the pairs (a_r, b_r) for $r = 1,2,\ldots,k$ partition $\{1,2,\ldots,k,k+2,k+3,\ldots,2k+1\}$, then the triples r, a_r+k, b_r+k for $r = 1,2,\ldots,k$ partition the set $\{1,2,\ldots,k,k+1,\ldots,2k,2k+2,2k+3,\ldots,3k+1\}$. So the existence of a (C,k)-system implies the existence of a cyclic $STS(6k+3)$.

Note. If a (D,k)-system exists, then the triples $r, a_r + k, b_r + k$ for $r = 1,2,\ldots,k$ is a solution to Heffter's Second Difference Problem. This is because $b_r = a_r + r$ implies $b_r + k = (a_r + k) + r$ and, since the pairs (a_r, b_r) for $r = 1,2,\ldots,k$ partition $\{1,2,\ldots,k,k+2,k+3,\ldots,2k,2k+2\}$, then the triples $r, a_r + k, b_r + k$ for $r = 1,2,\ldots,k$ partition the set $\{1,2,\ldots,2k,2k+2,2k+3,\ldots,3k,3k+2\}$. Notice that the value $3k+2$ as a difference is the same as the difference $3k+1$ in K_{6k+3} (since $|3k+2|_{6k+3} = 3k+1$). So the existence of a (D,k)-system implies the existence of a cyclic $STS(6k+3)$.

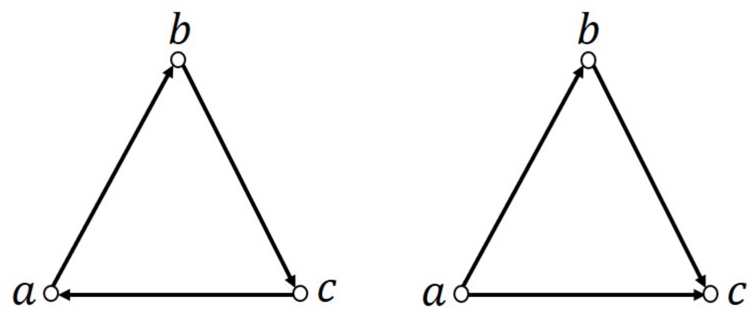
Note. Alex Rosa [13] in 1966 gave necessary and sufficient conditions for the existence of a (C, k)-system and a (D, k)-system, as follows.

Lemma T.5. A (C, k)-system exists if and only if $k \equiv 0$ or $3 \pmod 4$. A (D, k)-system exists if and only if $k \equiv 1$ or $2 \pmod 4$, $n \neq 1$.

Lemma T.6. There exists a cyclic $STS(n)$ for all $n \equiv 3 \pmod 6$, $n \neq 9$.

Theorem T.1. A $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$.

Note. We now turn our attention to digraphs and decompositions involving orientations of 3-cycles. There are two orientations of a 3-cycle:



Mendelsohn Triple

Directed Triple

We denote the Mendelsohn triple given here as $[a, b, c]_M = [b, c, a]_M = [c, a, b]_M$ and we represent the directed triple given here as $[a, b, c]_D$. We denote the complete digraph (that is, the digraph such that for every two vertices u and v there is exactly one arc with tail u and head v and there is exactly one arc with tail v and head u) on n vertices as D_n.

Definition. A decomposition of D_n into Mendelsohn triples is called a *Mendelsohn triple system* or order n, denoted $MTS(n)$. A decomposition of D_n into directed triples is called a *directed triple system* of order n, denoted $DTS(n)$.

Note. In 1971, Nathan Mendelsohn gave necessary and sufficient conditions for the existence of a Mendelsohn triple system of order n (thus the name of them; he called them a "natural generalization of Steiner triple systems") [10]. Mendelsohn, along with S. Hung, gave necessary and sufficient conditions for the existence of a directed triple system or order n in 1973 [6].

Theorem T.2. A $MTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$.

Theorem T.3. A $DTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$.

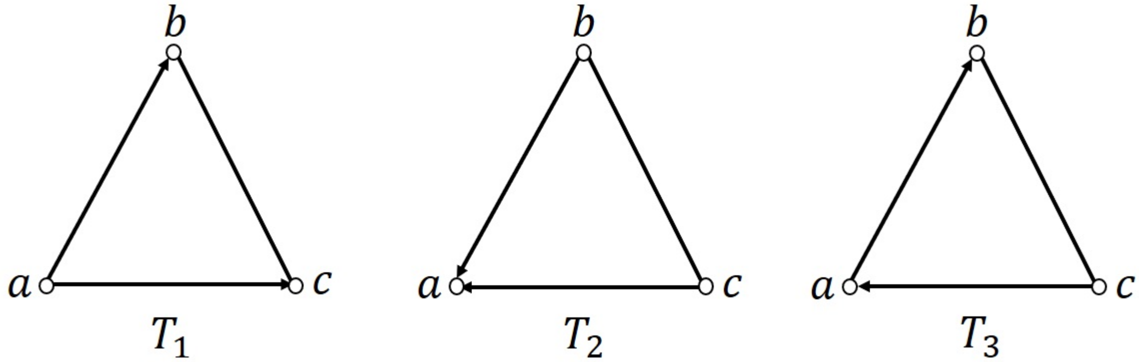
Note. Some Mendelsohn and directed triple systems are easy to construct using Steiner triple systems. If $n \equiv 1$ or $3 \pmod 6$ then a $STS(n)$ exists. We can take the triples of a $STS(n)$ and replace each triple $[a, b, c]$ with *either* the two Mendelsohn triples $[a, b, c]_M$ and $[c, b, a]_M$ *or* the two directed triples $[a, b, c]_D$ and $[c, b, a]_D$. This gives a $MTS(n)$ and a $DTS(n)$, respectively. This technique does not cover the cases of $n \equiv 0$ or $4 \pmod 6$.

Note. Variants of Mendelsohn and directed triple systems are the ideas of a hybrid triple system and an oriented triple system. Oriented triple systems (also called "ordered triple systems") were introduced by Curt Lindner and A. P. Street in 1984 [9]. Hybrid triple systems were introduced by Charlie Colbourn, W. R. Pulleyblank, and Alex Rosa in 1989 [1] and simple direct constructions were given by Katherine Heinrich in 1991 [5].

Definition. A *c-hybrid triple system* of order n, denoted $HTS(n)$, is a decomposition of the complete digraph D_n into c Mendelsohn triples and $v(v-1)/3-c$ transitive triples. An *oriented triple system* (also called an *ordered triple system* of order n), denoted $OTS(n)$, is a c-$HTS(v)$ where c is any of the values $0, 1, \ldots, v(v-1)/3$.

Theorem T.4. A c-$HTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$, and $c \in \{0, 1, 2, \ldots, n(n-1)/3 - 2, n(n-1)/3\}$, or $n = 6$ and $c \in \{0, 1, 2, \ldots, 8\}$. An $OTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$.

Note. Recall that a mixed graph consists of a vertex set, an edge set, and an arc set. The complete mixed graph on n vertices, denoted M_n, has for each pair u and v of distinct vertices, an edge joining u and v, an arc from u to v, and an arc from v to u. So M_n has twice as many arcs as edges. We can therefore extend the ideas of Steiner triple systems, Mendelsohn triple systems, and directed triple systems to mixed graphs. There are three distinct partial orientations of a 3-cycle which have twice as many arcs as edges:



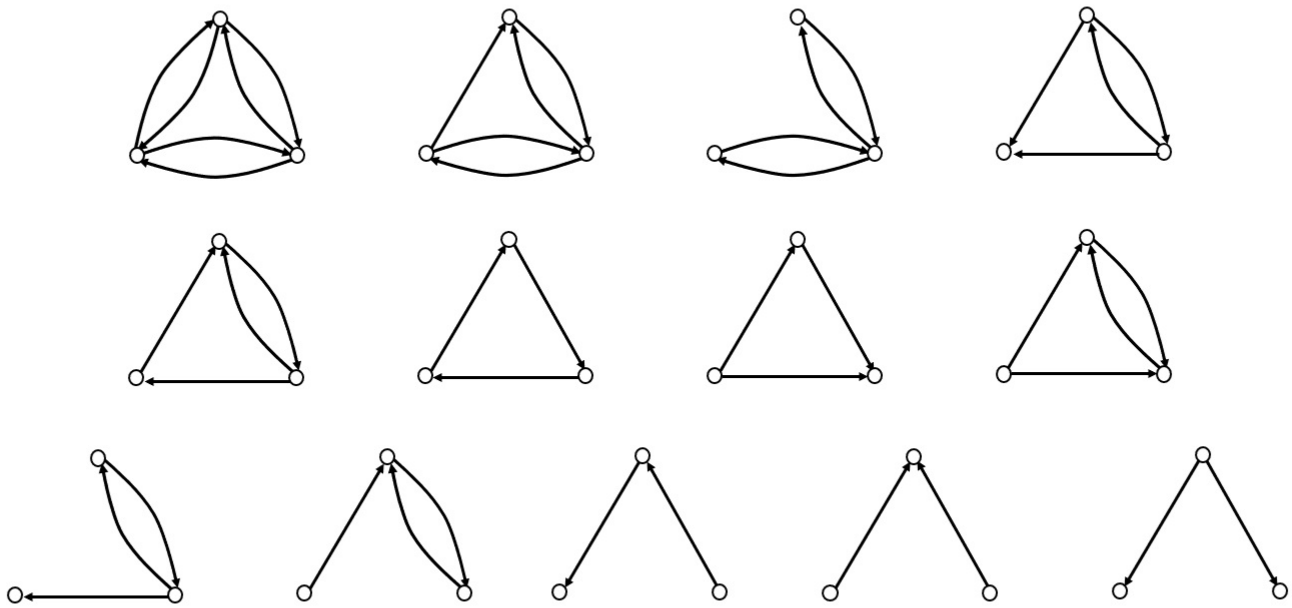
We call these *mixed triples* and denote mixed triple T_i as $[a, b, c]_i$ where $i \in \{1, 2, 3\}$. Mixed triple systems were introduced in 1999 by Robert "Dr. Bob" Gardner [2].

Definition. A decomposition of the complete mixed graph on n vertices into copies of T_i is a T_i-*mixed triple system*, where $i \in \{1, 2, 3\}$.

Theorem T.5. A T_i-mixed triple system of order n exists for each $i \in \{1, 2, 3\}$ if and only if $n \equiv 1 \pmod 2$, except in the cases $n \in \{3, 5\}$ when $i = 3$.

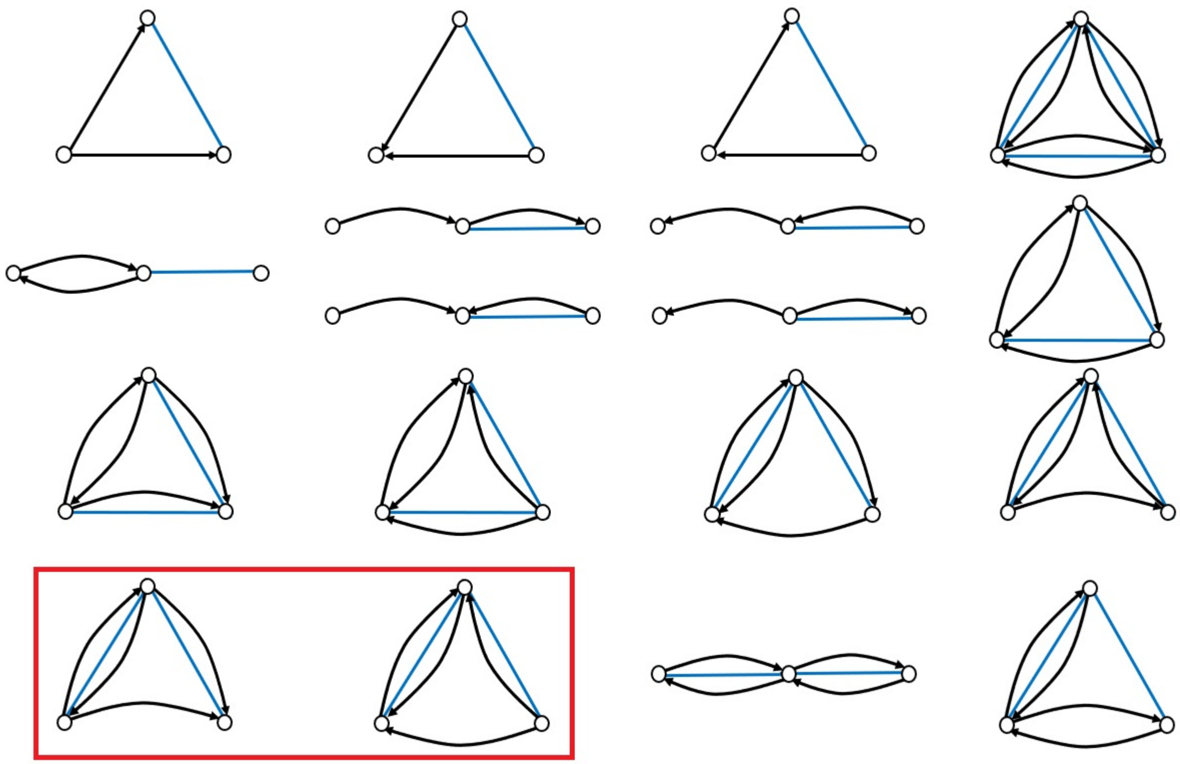
Note. Along the lines of a hybrid triple system in the setting of digraphs, we could consider hybrid mixed triple systems. This would be a decomposition of the complete mixed graph into m_1 copies of T_1, m_2 copies of T_2, and m_3 copies of T_3 where $m_1 + m_2 + m_3 = n(n-1)/2$. To date, nothing has been done in this direction (to my knowledge...), making this an **open problem**.

Note. In 1986 Alan Hartman and Eric Mendelsohn considered all (strict) digraphs on three vertices (of which there are 13, up to isomorphism):



They then defined 13 new types of "triple systems" in terms of decompositions of D_n into copies of each of these 13 digraphs. They gave necessary and sufficient conditions for the existence of each such triple system in their paper "The Last of the Triple Systems" [3].

Note. In 2009, ETSU graduate student Ernest Jum considered all (simple/strict) mixed graphs on three vertices with (like the complete mixed graph) twice as many arcs as edges (of which there are 18, up to isomorphism) in his master's thesis "The Last of the Mixed Triple Systems" [7]:



Mr. Jum gave necessary and sufficient conditions for the existence of a decomposition for the complete mixed graph into each of the triples, with the exception of the two mixed graphs outlined in red here (which are converses of each other), making this an **open problem**.

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