Chapter 14. Vertex Colourings
Study Guide

The following is a brief list of topics covered in Chapter 14 of Bondy and Murty's *Graph Theory*, Graduate Texts in Mathematics 244 (Springer, 2008). This list is not meant to be comprehensive, but only gives a list of several important topics. You should also carefully study the proofs given in class and the homework problems.

Section 14.1. Chromatic Number.
k-vertex colouring, proper colourings, k-colourable graph, colour class, chromatic number $\chi(G)$, k-chromatic, lower bound on $\chi(G)$ (equation 14.1), chromatic number and clique number (Note 14.1.B), finding chromatic numbers is an $\mathcal{NP}$-hard problem, the Greedy Colouring Heuristic (Heuristic 14.3), $\chi(G) \leq \Delta + 1$ (Lemma 4.1.A), classification of graphs for which $\chi(G) \neq \Delta + 1$ (Theorem 14.4, Brooks' Theorem), vertex colouring and chromatic number of a digraph, directed paths in digraphs (Theorem 14.5, the Gallai-Roy Theorem), k-optimal path partition, partial k-colouring, orthogonality of a path partition and a path k-colouring, the Path Partition Conjecture (Conjecture 14.6).

Section 14.2. Critical Graphs.
Colour critical graph, k-critical graph, in a k critical graph $\delta \geq k-1$ (Theorem 14.7), S-components of a graph, colourings of S-components that agree on S, no critical graph has a clique cut (Theorem 14.8), critical graphs are nonseparable (Corollary 14.9), Type 1 and 2 $\{u, v\}$-components of a graph.

Section 14.3. Girth and Chromatic Number.
Graphs with large girth and chromatic number (Theorem 14.11 of Erdős), probabilistic method of proof, triangle-free k-chromatic graphs (Theorem 14.12).

Section 14.4. Perfect Graphs.
Perfect and imperfect graphs, the Perfect Graph Theorem (Theorem 14.13), classification of perfect graphs (Theorem 14.14), stable sets and clique numbers (Proposition 14.15 and Lemma 14.16), illustration of the proof of Lemma 14.16 based on C_7, the Strong Perfect Graph Theorem (Theorem 14.19).

Section 14.5. List Colouring.

List of colours, list colouring of a graph, L-colouring and L-colourable, differences in k-chromatic and L-colourable (Figure 14.9), k-list colourable, list chromatic number $\chi_L(G)$, kernels and L-colourings of digraphs (Theorem 14.20), every graph is $(\Delta + 1)$-list-colourable (Corollary 14.21), interval graphs have list chromatic number ω (Corollary 14.22).

Section 14.6. The Adjacency Polynomial.

Adjacency polynomial (graph polynomial), the use of fields in colourings (Note 14.6.A), monomials in an adjacency polynomial, Vandermonde matrix/Vandermonde determinant, the sign of the orientation of a graph, the weight of a sequence of nonnegative integers, the Hilbert Hullstellensatz, list colourings based on lists L_i of $d_i + 1$ colours (Proposition 14.23), the Combinatorial Nullstellensatz (Theorem 14.24), orientations of G and list colourings (Corollary 14.25).

Section 14.7. The Chromatic Polynomial.

The number of k-colourings of G $C(G, k)$, distinct colourings of a graph, equal k-colourings, $C(G, k)$ for empty graphs and complete graphs (Note 14.7.A), the recursion formula for $C(G, k)$ (equation 14.5), the chromatic polynomial its properties and the recursion formula (Theorem 14.26), finding $P(G, x)$ using the recursion formula (equation 14.6), chromatic roots, Sokal's result on the density of chromatic roots in $\mathbb{C}$.

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