Supplement. The Four-Color Theorem: A History, Part 1

Note. Simply put, the Four Color Theorem states that the countries of any map can be colored with four colors such that any two countries which share a border (not just a point on their border, but a segment of the border; notice Utah and New Mexico below). As an example, the 48 contiguous states of the United States can be colored as follows:

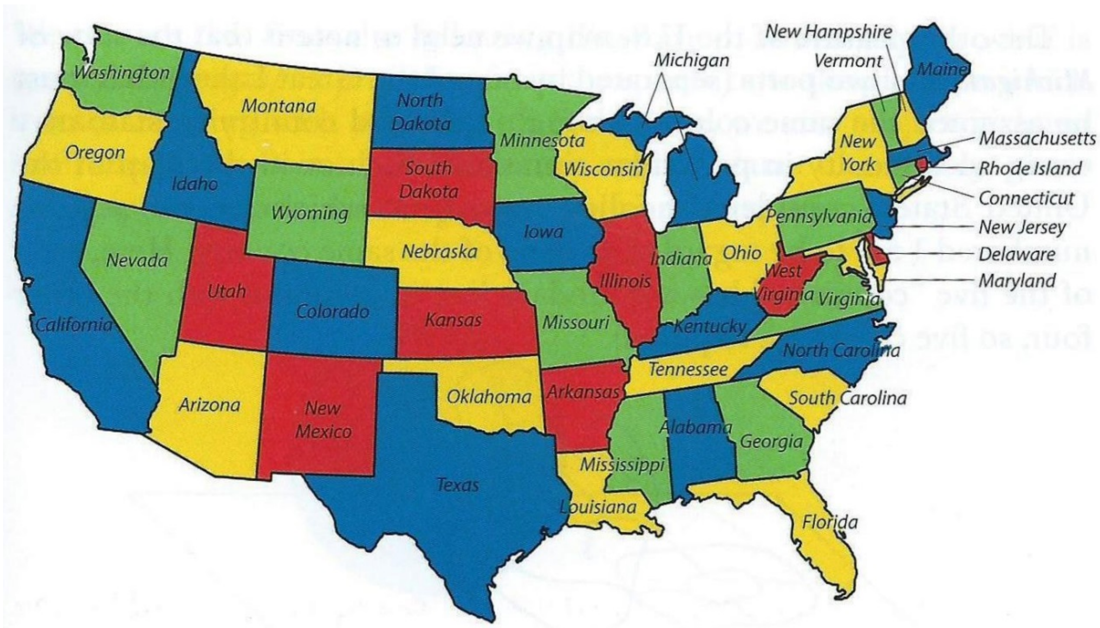
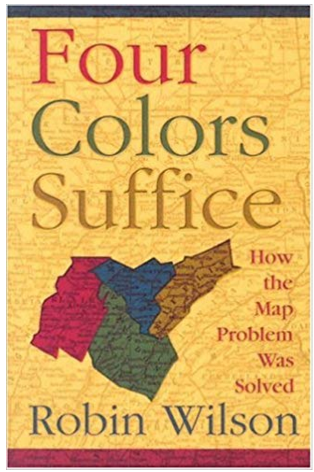
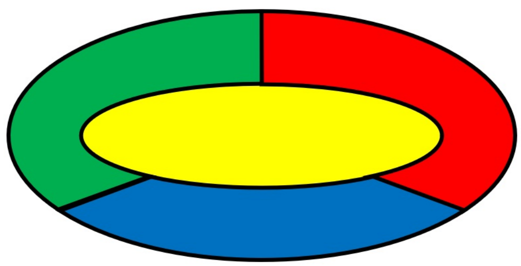


Image from University of Illinois Urbana-Champaign Department of Mathematics webpage (accessed 12/17/2022)

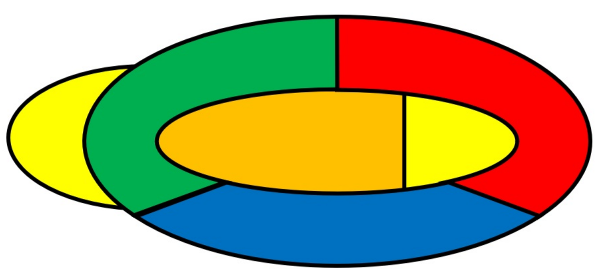
This supplement is largely historical, but will overlap with some of the mathematical material of Chapter 11 ("The Four-Colour Problem") and Chapter 15 ("Colourings of Maps") of J. A. Bondy and U. S. R. Murty's *Graph Theory*, Graduate Texts in Mathematics #244 (Springer, 2008). The historical material is (unless stated otherwise) from Robin Wilson's *Four Colors Suffice*, Princeton University Press (2002)—this supplement is largely a "book report" on *Four Colors Suffice* and we will reference it as "Wilson."



Note FCT.A. It is easy to see by example that *at least* four colors are required. Consider a ring of three countries surrounding a fourth country, as follows:



Since each country is adjacent to the other three, then we have that four colors are needed for this map. The small European country of Luxembourg is surrounded by its neighbors France, Belgium, and Germany in this way (see alearningfamily.com for a map; accessed 12/18/2022). We introduce another convention. We require that countries are connected regions.

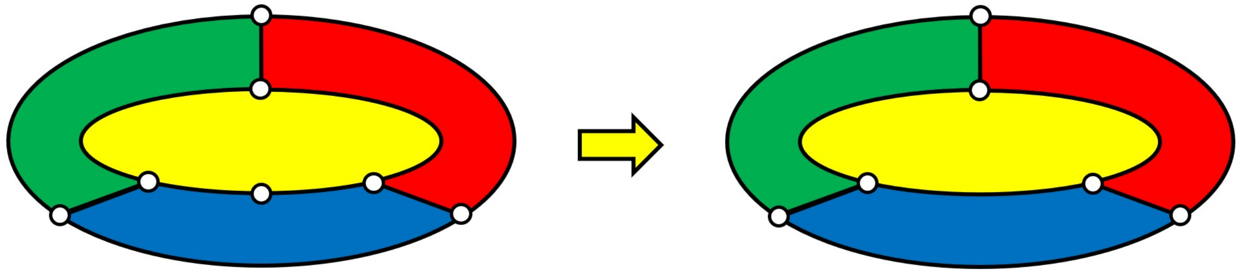


Above, the yellow country comes in two components. Each country shares a border with the other four, so that five colors are needed. There are currently countries in the real world that are not connected, however. The Naxçivan region of Azerbaijan is separated from the rest of the country (due to conflict with its neighbor, Armenia). Of course Alaska is seperated from the "lower 48" in the U.S.

Image from Worldometers.info webpage (accessed 12/17/2022)

If we allow disconnected countries, then for n countries we could simply put in each country an "island" (these are called *enclaves*) of the $n-1$ other countries, and then n colors would be required. This then destroys the map coloring problem.

Note FCT.B. As we converted the map coloring problem over to the graph theoretic setting, we impose one final convention. We require that at least three borders meet at each meeting point. If there are only two borders at a meeting point (below left), then we can delete that point with no impact on the coloring (below right).



Note. The first written record of the four color *problem* is a letter dated October 23, 1852. In this letter, Augustus De Morgan of University College London wrote Irish mathematician William Rowan Hamilton of Dublin; the two corresponded regularly.

William R. Hamilton

(June 27, 1806—March 18, 1871)

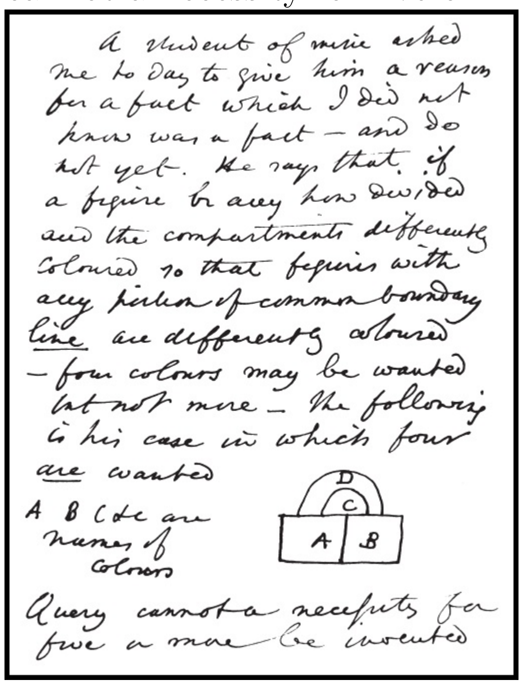
Augustus De Morgan

(August 4, 1805–September 2, 1865)

Images from the MacTutor History of Mathematics Archive biographies of Hamilton and De Morgan (accessed 12/18/2022)

The letter read in part (Wilson, page 18):

"A student of mine [Frederick Guthrie, brother of Francis] asked me today to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured—four colours may be wanted, but not more—the following is the case in which four colours are wanted. Query cannot a necessity for five or more be invented..."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A B C &c are
names of
colours

Query cannot a necessity for
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This close-up image of the letter is from Wilson; Chapter 2 of Wilson is available on the Princeton University Press webpage; a copy of the complete letter is available on Wikipedia's Four Color Theorem (accessed 12/18/2022). The little image with regions "coloured" A, B, C, and D show that at least four colors are necessary.

Note. The earliest known appearance of the four color problem in print is in the April 14, 1860 issue of the literary journal Athenaeum. In a review of William

Whewell's book *The Philosophy of Discovery, Chapters Historical and Critical*, the reviewer (anonymous at the time, but widely believed to have been De Morgan) outlined the problem and claimed that it is familiar to cartographers (though Wilson on page 2 states: "Rather surprisingly, perhaps, the four-colour problem has been of little importance for mapmakers and cartographers."). He also observed that it had not been proved. This book review and commentary brought the problem to the attention of the American mathematician and logician Charles Sanders Peirce (pronounced 'purse'; 1839-1914) who brought it to the attention of the American mathematical community at Harvard University. Arthur Cayley (1821–1895) was interested in the problem, asked about it at an 1878 meeting of the London Mathematical Society and published a short note on it in the April 1879 issue of the *Proceedings of the Royal Geographic Society* in which he mentioned that he had made little progress on a proof.

Note. In around 1840, August Ferdinand Möbius (1790–1868) posed "the problem of the five princes," which has a superficial similarity to the four color problem. In 1885, German geometer Richard Baltzer published an article in *Mathematisch-physiche Classe* in which he solved the five princes problem and claimed that the four color problem follows from his five princes solution. Isabel Maddison of Bryn Mawr College published "Note on the History of the Map-Coloring Problem" in *Bulletin of the American Mathematical Society*, **37**(7), 257 (1897). It is available online at the ProjectEuclid.org website (accessed 12/18/2022). The AMS journals are widely circulated, and this lead to the belief that Möbius was the first to state the four color problem. This claim appeared in the well-known history of math book *Mathematical Recreations*, 11th edition 1939, by Walter Rouse Ball, and again in *The Development of Mathematics*, 2nd edition 1945, by Eric Temple Bell. H.S.M. Coxeter published "The Four-Color Map Problem, 1840–1890," *The Mathematics Teacher*, **52**(4), 283–289 (1959) (available online at the JSTOR website, accessed 12/18/2022). In this, Coxeter corrected the erroneous history and since then Francis Guthrie has been recognized as the originator of the four color problem, as mentioned in De Morgan's letter to Hamilton.

Note. In Section 10.3. Euler's Formula, we briefly discussed the history of Euler's formula and the involvement of René Descartes (1596–1650), Leonhard Euler (1707–1783), and Adrien-Marie Legendre (1752–1833). Recall that Legendre was the first to give a rigorous proof in his 1794 *Eléments de geométrie*. In Theorem 10.19 of Section 10.3, we stated Euler's formula for a plane graph G: $v(G) - e(G) + f(G) = 2$. We used Euler's formula to prove the "Only Five Neighbors Theorem" in Corollary 10.22 of Section 10.3:

> **Only Five Neighbors Theorem.** Every map has at least one country with five or fewer neighbors.

If we represent a map with a graph G, then we can consider the dual G^* of G, as defined in Section 10.2. Duality. The Only Five Neighbors Theorem can then be stated (as it is in Corollary 10.22) as: "Every simple planar graph has a vertex of degree at most five."

Note. In a "cubic map" there are exactly three boundary lines at each meeting point. We apply Euler's formula to prove the following, which we will need when discussing "discharging" of a map later.

The Counting Formula for Cubic Maps. In a cubic map determined by plane graph G, let C_k denote the number of countries with k sides, where $k \geq 2$. Then we have:

$$4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C + 8 - 3C_9 - \cdots - (k-6)C_k - \cdots = 12.$$

Proof. The number of faces is $f(G) = C_2 + C_3 + C_4 + \cdots$. Because each boundary line lies on the boundary of two countries, we have that the total number of boundary lines (or edges) is $2e(G) = 2C_2 + 3C_3 + 4C_4 + 5C_5 + \cdots$, or

$$e(G) = C_2 + \frac{3}{2}C_3 + 2C_4 + \frac{5}{2}C_5 + 3C_6 + \frac{7}{2}C_7 + \cdots.$$

Now three boundary lines meet as each vertex, so $3v(G) = 2e(G)$ or $3v(G) = 2e(G) = 2C_2 + 3C_3 + 4C_4 + 5C_5 + \cdots$, and hence

$$v(G) = \frac{2}{3}C_2 + C_3 + \frac{4}{3}C_4 + \frac{5}{3}C_5 + 2C_6 + \frac{7}{3}C_7 + \cdots.$$

By Euler's formula we now have

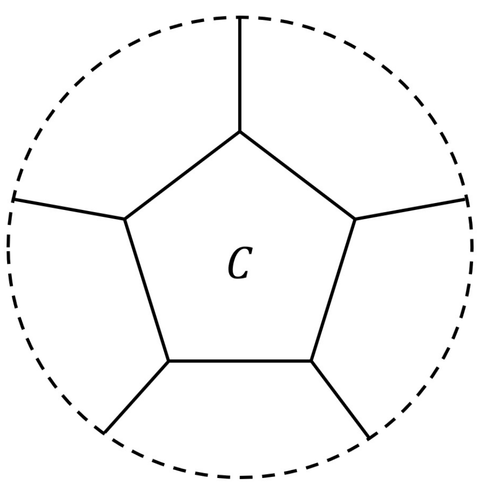
$$\begin{aligned} 2 &= v(G) - e(G) + f(G) = \left(\frac{2}{3}C_2 + C_3 + \frac{4}{3}C_4 + \frac{5}{3}C_5 + 2C_6 + \frac{7}{3}C_7 + \cdots\right) \\ &\quad - \left(C_2 + \frac{3}{2}C_3 + 2C_4 + \frac{5}{2}C_5 + 3C_6 + \frac{7}{2}C_7 + \cdots\right) + (C_2 + C_3 + C_4 + \cdots) \\ &= C_2(1 - 1 + 2/3) + C_3(1 - 3/2 + 1) + C_4(1 - 2 + 4/3) \\ &\quad + C_5(1 - 5/2 + 5/3) + C_6(1 - 7/2 + 7/3) + \cdots \\ &= \frac{2}{3}C_2 + \frac{1}{2}C_3 + \frac{1}{3}C_4 + \frac{1}{6}C_5 + 0C_6 - \frac{1}{6}C_7 - \cdots. \end{aligned}$$

Multiplying both side by 6 yields the result. ∎

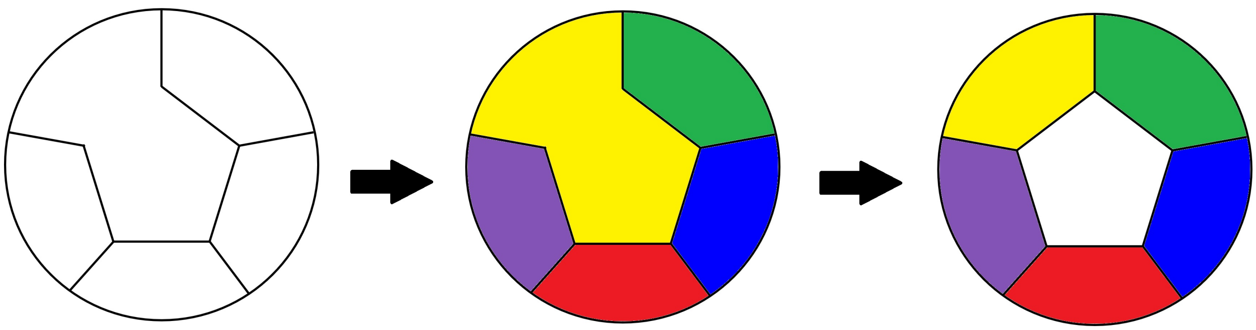
Note. Next, we give a proof of the Six-Color Theorem; that is, every map can be colored with six colors such that neighboring countries are colored differently. We present the proof based on Wilson's explanation given on his pages 71 and 72 (so it is a little informal). The proof uses the Only Five Neighbors Theorem and illustrates the approach taken in attempts to prove the Four Color Theorem.

The Six-Color Theorem. Every map can be colored with six colors such that neighboring countries are colored differently.

Proof. ASSUME that there exist maps that cannot be six colored. Among these maps (which require seven or more colors), consider one with the smallest possible number of countries. Such a map is a minimal counter-example (Wilson uses the term *minimal criminal* in this context). This idea is given in Bondy and Murty in Note 15.2.A of Section 15.2. The Four-Colour Theorem. Since the map is minimal, then any map with fewer countries can be colored with six colors. By the Only Five Neighbors Theorem, there is a country C with five or fewer neighbors:

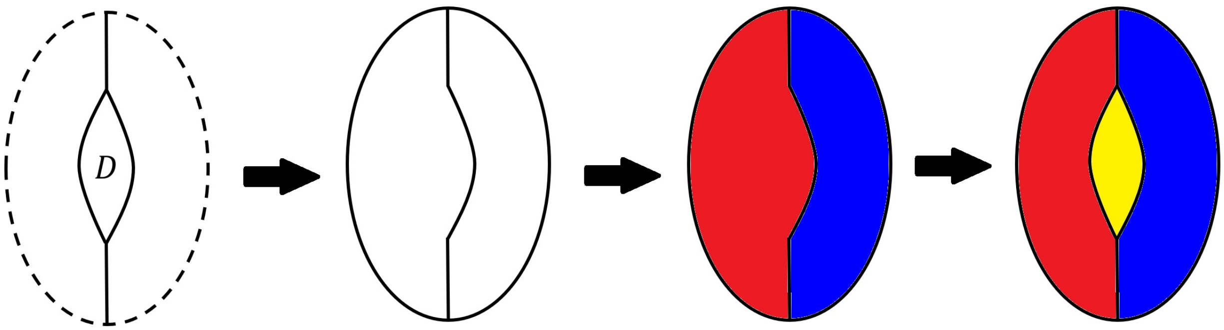


First, we remove a boundary line from C so that the number of countries is reduced by one (below left). Since there are fewer countries than in the minimal counterexample, then the new map can be colored with six colors, say red, blue, green, yellow, purple, and white (below center). Finally we put the removed boundary line back to reinstate country C. Since six colors are available and country C has only five neighbors, then one color is available to properly color country C (below right).

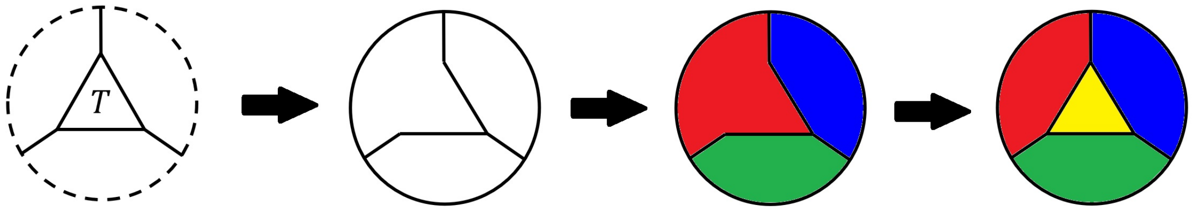


But this CONTRADICTS the fact that the map cannot be six colored. So the assumption that there exists a map that cannot be six colored is false, and hence every map can be six colored. ∎

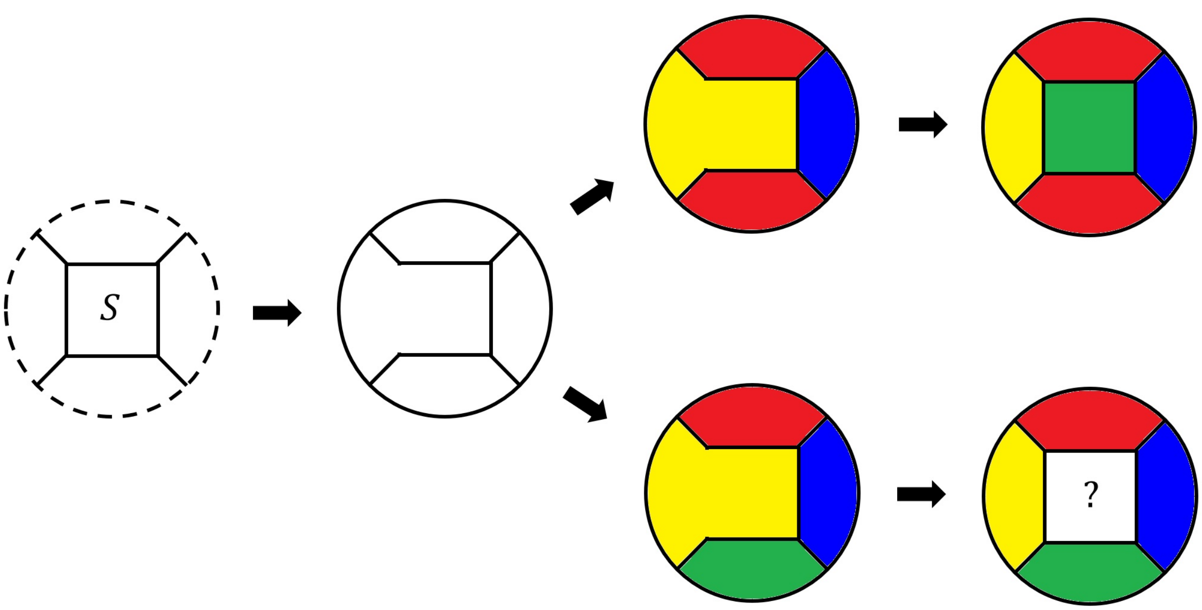
Note FCT.C. We can follow the ideas in the proof of the Six Color Theorem to show that a minimal counterexample to the Four Color Theorem cannot contain a two-sided country. Wilson calls such countries "digons." This fact, combined with the fact that a minimal counterexample cannot contain a one-sided country (since such a country would have to lie inside of another country and could simply be colored something different from the color of the country surrounding it). This is why Bondy and Murty only consider simple graphs in connection with maps. In the following map we have a two-sided country D ("digon," left), which we assume is a minimal counterexample to the Four Color Theorem. We remove one boundary of the digon (second), yielding a map with less countries and hence a map that can be four colored. Four coloring the map implies that the two countries given (third) must be of different colors. We put the removed boundary back (right) and then we can color the digon D either of the colors not used on the two countries bordering D. But this shows that such a map cannot be a minimal counterexample to the Four Color Theorem.



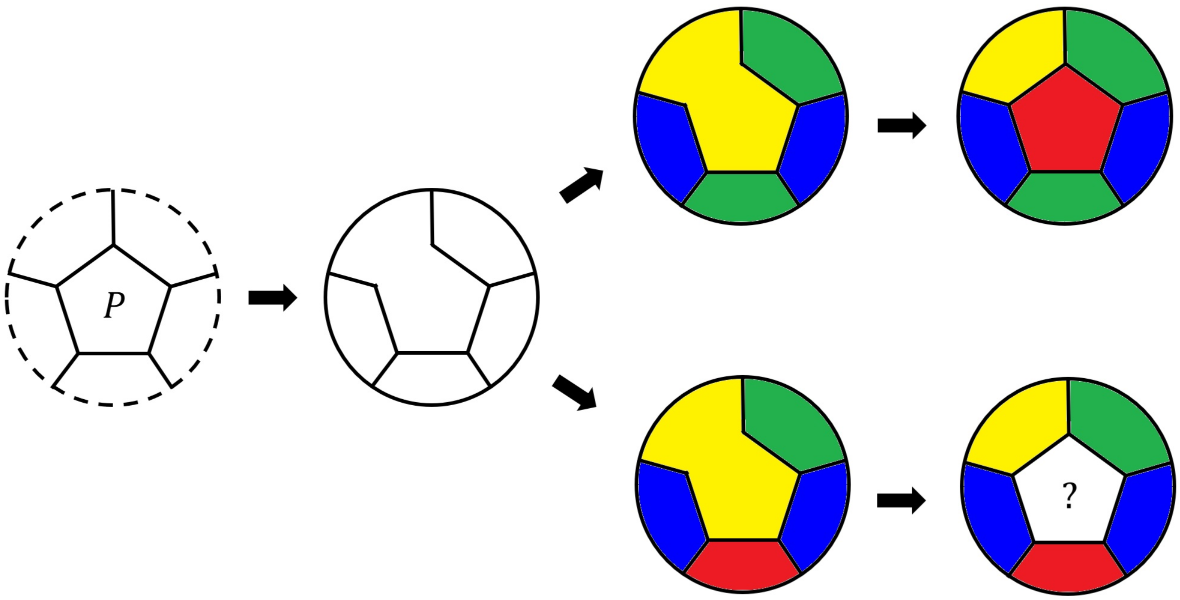
Note FCT.D. We apply the same technique to a minimal counterexample to the Four Color Theorem to show that such a counterexample cannot contain a three-sided county (i.e., a "triangle"). This is stated in Proposition 15.2(iii) of Bondy and Murty's of Section 15.2. The Four-Colour Theorem (the dual of the map is being discussed there, so the result is stated as: A smallest counterexample to the Four Color Theorem has no vertex of degree less than four). We start with an assumed minimal counterexample to the Four Color Theorem containing a triangle (left). We remove one boundary of the triangle, yielding a map with less countries (second) and hence a map that can be four colored (third). We put the removed boundary back (right) and then we can color the triangle T with the appropriate fourth color, contradicting the assumption of a minimal counterexample containing a triangle.



Note FCT.E. Filled with unjustified optimism, we might attempt to show that no minimal counterexample for the Four Color Theorem exists by arguing that such a counterexample cannot contain an n-sided country where $n \in \mathbb{N}$ (maybe arguing inductively?). However, this approach has reached the end of its usefulness. If we consider the same technique as applied to a four-sided country, then we potentially have a problem (below, lower right). When we remove a boundary from the square, we know that there is then a four coloring (third), but we don't know in detail how the colors may be distributed. They may be distributed in a way that does not permit a four coloring of the original make once put the removed boundary back.



The same problem arises in considering a five-sided country (below).



In the problem case, the green, red, and blue countries could be differently colored and the problem still exists. For example, the lower left blue country could be green.

Note. Quoting from Wilson's page 73:

> "We come now to the most famous fallacious proof in the whole of mathematics—the purported solution of the four-colour problem by the London barrister and amateur mathematician Alfred Bray Kempe (pronounced 'kemp'). It is unfortunate that he is remembered mainly for his flawed proof, since he was a fine mathematician, highly regarded by his contemporaries. In Kempe's defence, his error was a subtle one that would remain undetected for eleven years, and his 'solution' embodies a number of original ideas that proved to be of the greatest importance in later work on the problem."

Alfred Bray Kempe (July 6, 1849–April 21, 1922)

Image from the MacTutor History of Mathematics Archive biography of Kempe (accessed 12/23/2022)

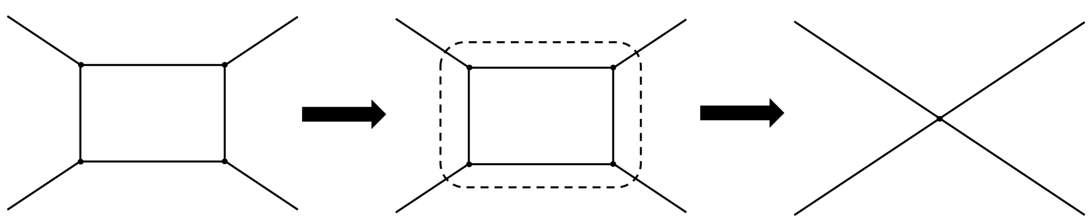
Kempe studied with Arthur Cayley to Trinity College. His interest in the coloring of maps was stimulated when he attended the 1878 meeting of the London Mathematical Society where Cayley asked questions about the coloring conjecture (as mentioned above). By June 1879, Kempe had his "proof" of the Four Color Theorem, a preview of which he published in the July 17, 1879 issue of *Nature*, **20** (507), 275. A copy is online in PDF on the nature.com website (accessed 12/23/2022). A full version of Kempe's proof appeared in "On the Geographical Problem of the Four Colours," *American Journal of Mathematics*, **2**(3), 193–200 (1879), available online at JSTOR website (accessed 12/23/2022). This paper is also reprinted in N. L. Biggs, E. K. Lloyd, and R. J. Wilsons *Graph Theory: 1736–1936*, Oxford University Press (1976). On February 26, 1880 Kempe published a simplified version of his solution in *Nature* in "How to Colour a Map with Four Colours," **21**(539), 399-400 (1880), available online on nature.com (accessed 12/23/2022). He also published a simplified version in the *Proceedings of the London Mathematical Society*, **10**, 229–231 (1878-79). In the simplified versions he corrected some minor errors from his original paper but left intact the fatal flaw.

Note FCT.F. Kempe's "proof" uses the Only Five Neighbors Theorem to describe a method for coloring any map. Wilson states it (on his pages 79 and 80) in six steps as:

1. Start with a country with five or fewer neighbors.

2. "Cover" this country with a patch of the same shape, but slightly larger.

3. Treat the patch as a point with the borders intersecting it in the original map as edges incident to it in the new map (this amounts to shrinking the country to a point).



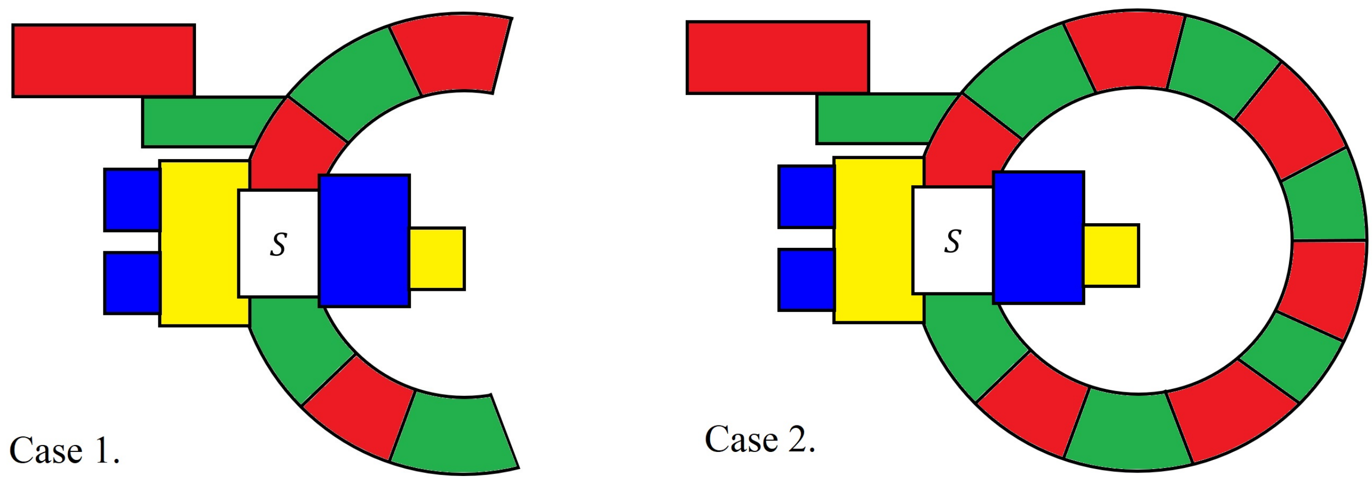
4. Repeat the first three steps iteratively until there is just one country left.

5. Color the single remaining country with any of the four colors.

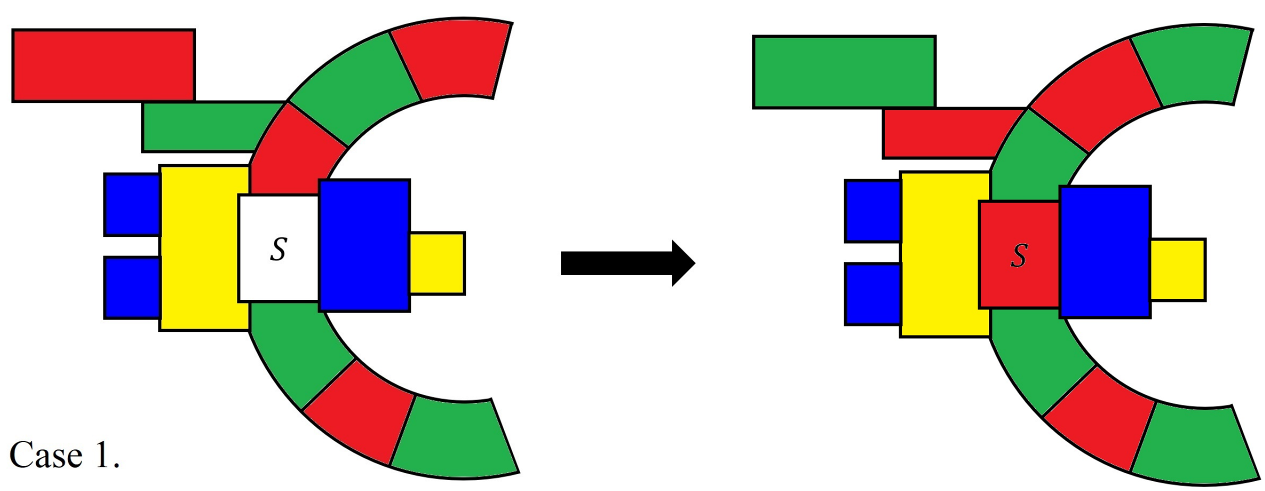
6. Reverse the process, stripping off the patches in reverse order, until the original map is restored. At each stage, color the restored country with any available color until the entire map is four-colored.

The problem that must be addressed is how to color countries as the patches are removed in such a way as to get a four coloring. Kempe's approach to this part of the problem is his most important and enduring contribution to the Four Color Theorem. The idea is interchange two colors as needed so that step 6 can be completed. This involves following a *Kempe chain* of countries and the interchanging of two colors is called *Kempe interchange*. This is the procedure used in Bondy and Murty's proof of Theorem 15.3 (in which it is shown that no country with four neighbors can be in a minimal counterexample to the Four Color Theorem; it's stated in terms of a degree four vertex, but this is because Bondy and Murty are considering the dual of the map).

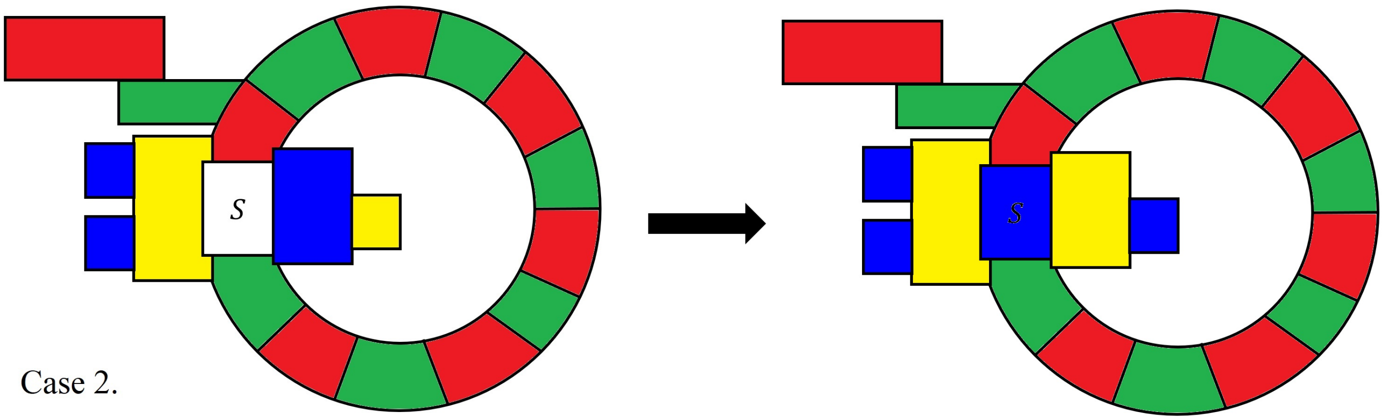
Note FCT.G. We now illustrate Kempe's technique in the case of a country with four sides (i.e., a square S), where the four neighboring countries use all four colors. In this case, no color is available for country S in a four coloring. Kempe then rearranges two of the colors in such a way that one of the four colors is available for country S. We discuss this in terms of red-green countries in the following two illustrations. On one side of S we have a red country and on the opposite of S we have a green country. We follow red country to adjacent green country(ies), then to adjacent red country(ies), etc. This gives a part of the map consisting only of red and green countries. We do the same by starting with the green country adjacent to S. This leads to two cases. In Case 1, the collection of red-green countries starting at the red country adjacent to S are separate from the red-green countries starting at the green country adjacent to S. In Case 2, the collections of red-green countries "link together" (in an informal use of the term "link").



In Case 1, we can *interchange* red and green colors in the red-green countries of the collection of such countries which start with the red country adjacent to S. Since these countries are not linked to the red-green countries starting with the green country adjacent to S, then no color conflict arises. We can then color S red.



In Case 2, the linking together of the red-green countries from the top of S to the bottom of S allows us to concentrate on the blue-yellow countries *inside* this connecting collection of countries (to the right of S in the picture). Now we interchange blue and yellow colors in this blue-yellow countries. Since these countries are not linked to the other blue-yellow countries (since these countries are enclosed by the ring of red-green countries), no color conflict arises. We can then color S blue.

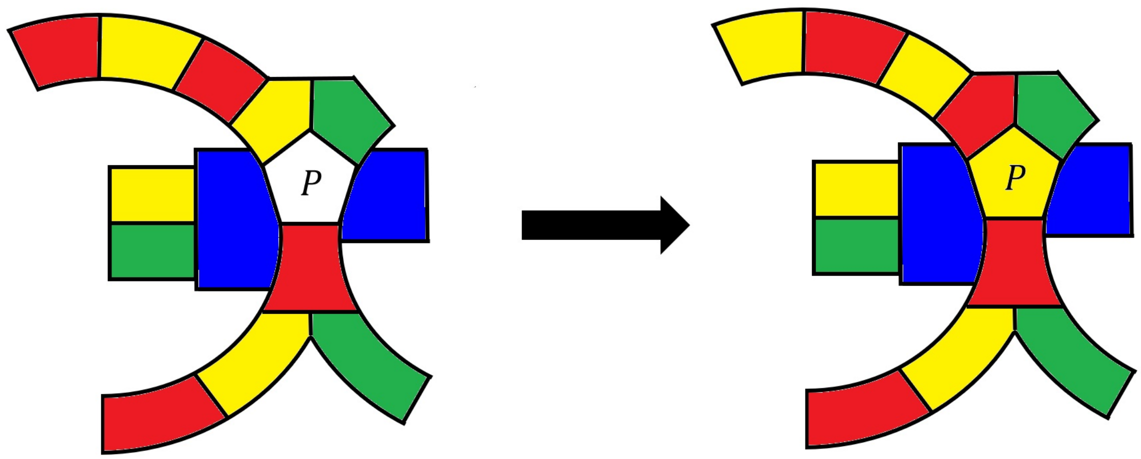


Case 2.

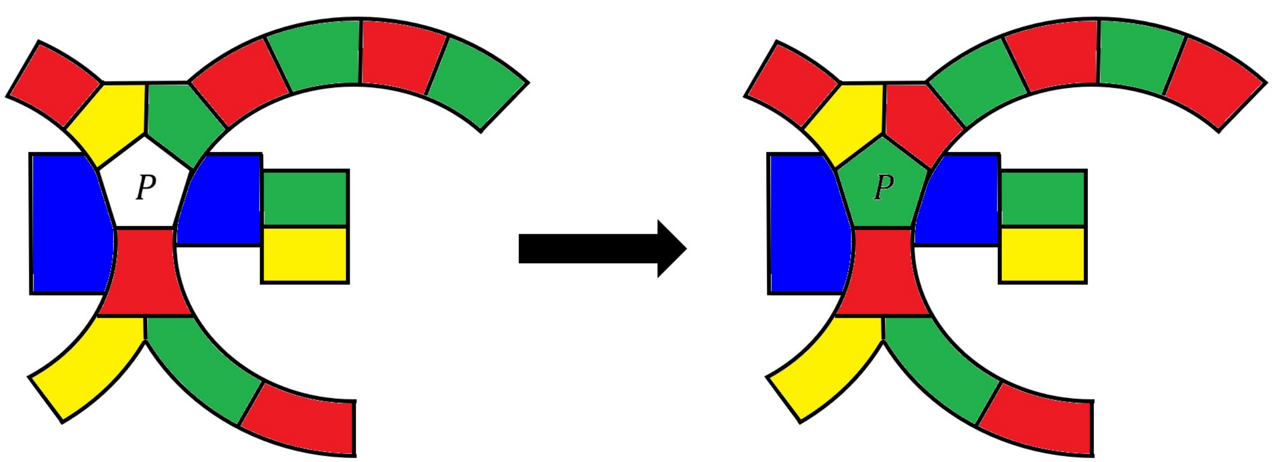
Note FCT.H. We now apply Kempe's technique to a minimal counterexample to the Four Color Theorem. ASSUME such an example contains a four-sided country. Since the map is a minimal counterexample, if we replace the four-sided country S with a single point using the "patch" technique of Kempe in Note FCT.F, then the resulting new map can be four colored. In this four coloring, if the four countries neighboring country S in the original map include three or less colors, then we can reintroduce country S and give it one of the available four colors. But this is a CONTRADICTION to the fact that the original map cannot be four colored. If the four countries neighboring country S in the original map include all four colors, then we can apply Kempe interchange to assign one of the four available colors. Again, this is a CONTRADICTION to the fact that the original map cannot be four colored. Therefore, no country with four neighbors can be in a minimal counterexample to the Four Color Theorem!

Note. At this stage, we know by the Only Five Neighbors Theorem that every map has at least one country with five or fewer neighbors, that a minimal counterexample to the Four Color Theorem cannot contain a two-sided country (by Note FCT.C), that a minimal counterexample cannot contain a three-sided country (by Note FCT.D), and that a minimal counterexample cannot contain a four-sided country (by Note FCT.H). Therefore, if we can show that a minimal counterexample cannot contain a five-sided country, then a minimal counterexample to the Four Color Theorem cannot exist, so that no counterexample exists. In this way, we would have a proof of the Four Color Theorem. This is the approach that Alfred Kempe takes and it is in his attempt to show that Kempe chains and interchange can be used to show that a minimal counterexample to the Four Color Theorem can, in fact, be four colored. It is in this argument that he makes his infamous mistake.

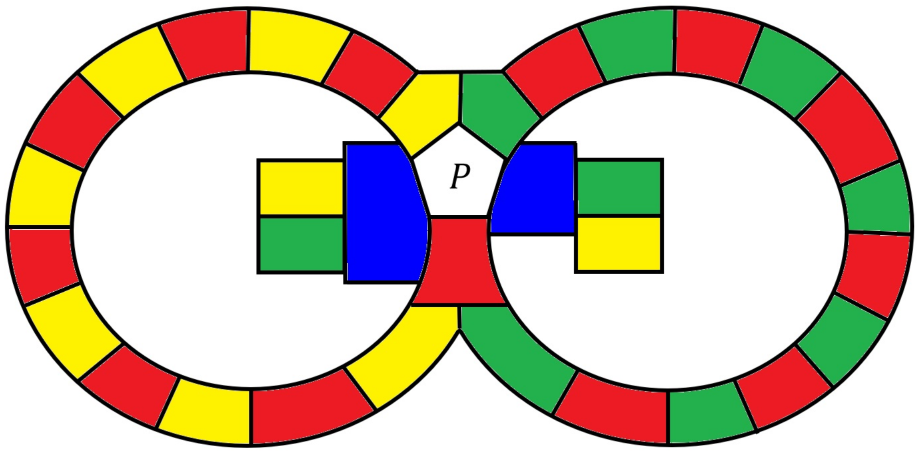
Note FCT.I. We next illustrate Kempe's attempt to apply his technique in the case of a country with five sides (i.e., a pentagon P), where the five neighboring countries use all four colors. In this case, no color is available for country P in a four coloring. First consider the yellow country and the red country neighboring P. If the chain of red-yellow countries starting with the yellow country neighboring P is not linked to the chain of red-yellow countries starting with the red country neighboring P, then we can interchange yellow countries and the red countries of the chain starting with the yellow neighbor of P (the upper red-yellow chain in the following diagram). As in the case of a four-sided country, no color conflict arises. The pentagon P can then be colored yellow.



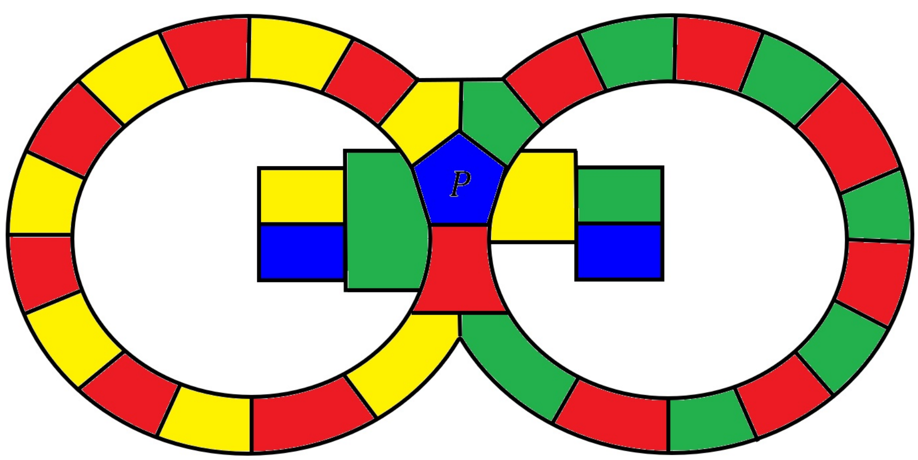
Similarly, if the chain of red-green countries starting with the green country neighboring P is not linked to the chain of red-green countries starting with the red country neighboring P, then we can interchange red countries and green countries of the chain starting with the green neighbor of P (the upper red-green chain in the following diagram). The pentagon P can can then be colored green.



We are now left with the cases involving the upper and lower red-yellow chains linked *and* the upper and lower red-green chains linked, as in the following diagram.



In this diagram, the blue-yellow chain to the right of pentagon P is separated by the red-green chain from the blue-yellow part to the left of P (and the blue-yellow part exterior to the red-green chain). We can interchange blue countries and yellow countries which are interior to the red-green chain with no color conflict arising. Similarly, the blue-green chain to the left of pentagon P is separated by the red-yellow chain from the blue-green part to the right of P (and the blue-green part exterior to the red-yellow chain). We can interchange blue countries and green countries which are interior to the red-yellow chain with no color conflict arising. Combining these we get the following and see that pentagon P can be colored blue.



It is subtle, but Kempe's error lies in this double swapping of colors.

Note FCT.J. We again apply Kempe's technique to a minimal counterexample to the Four Color Theorem. ASSUME such an example contains a five-sided country. Since the map is a minimal counterexample, if we replace the five-sided country P with a single point using the "patch" technique of Kempe in Note FCT.F, then the resulting new map can be four colored. In this four coloring, if the five countries neighboring country P in the original map include three or less colors, then we can reintroduce country P and give it one of the available four colors. But this is a CONTRADICTION to the fact that the original map cannot be four colored. If the five countries neighboring country P in the original map include all four colors, then we can apply Kempe interchange to assign one of the four available colors. Again, this is a CONTRADICTION to the fact that the original map cannot be four colored. Therefore, no country with five neighbors can be in a minimal counterexample to the Four Color Theorem! We have now seen, by the Only Five Neighbors Theorem, that every map has at least one country with five or fewer neighbors, but that a minimal counterexample to the Four Color Theorem cannot contain a two-sided country (by Note FCT.C), a three-sided country (by Note FCT.D), a four-sided country (by Note FCT.H), nor a five-sided country (by this Note). Therefore, no minimal counterexample to the Four Color Theorem exists, so that every map must satisfy the Four Color Theorem. **This is Kempe's (erroneous) proof of the Four Color Theorem.**

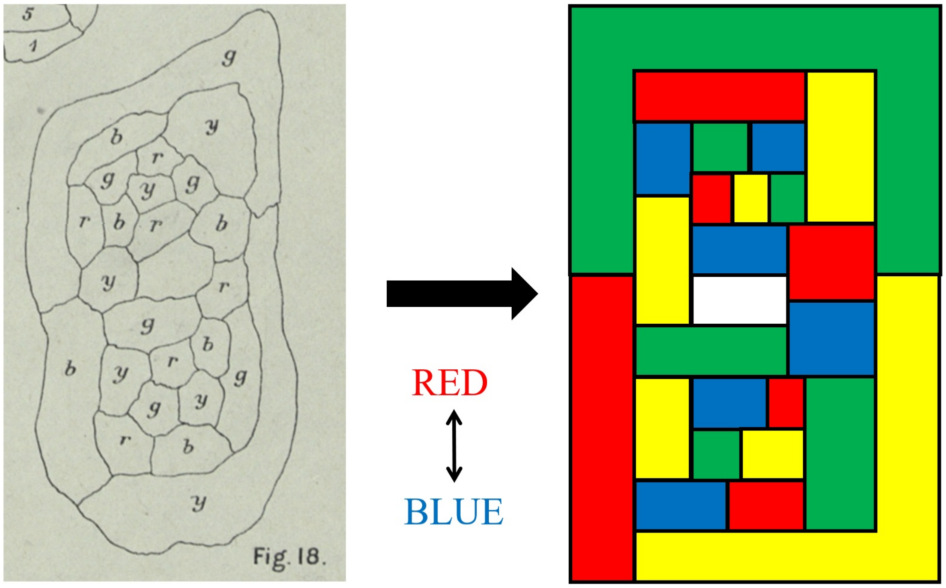
Note. Kempe's alleged proof of the Four Color Theorem was widely accepted. Kempe was nominated as a Fellow of the Royal Society on November 24, 1879 and was elected on June 2, 1881. See Wilson's page 95.

Note. The error in Kempe's paper was found by Percy Heawood (pronounced 'haywood'), a mathematics lecturer at the Durham Colleges (now Durham University). He published his findings in Map-Colour Theorem *The Quarterly Journal of Pure and Applied Mathematics*, **24**, 332–338 (1890). A copy of volume 24 of the journal is online at Goettingen Digitization Center webpage and Heawood's paper is available there (the figures referred to in the paper appear at the end of the journal; accessed 12/25/2022).

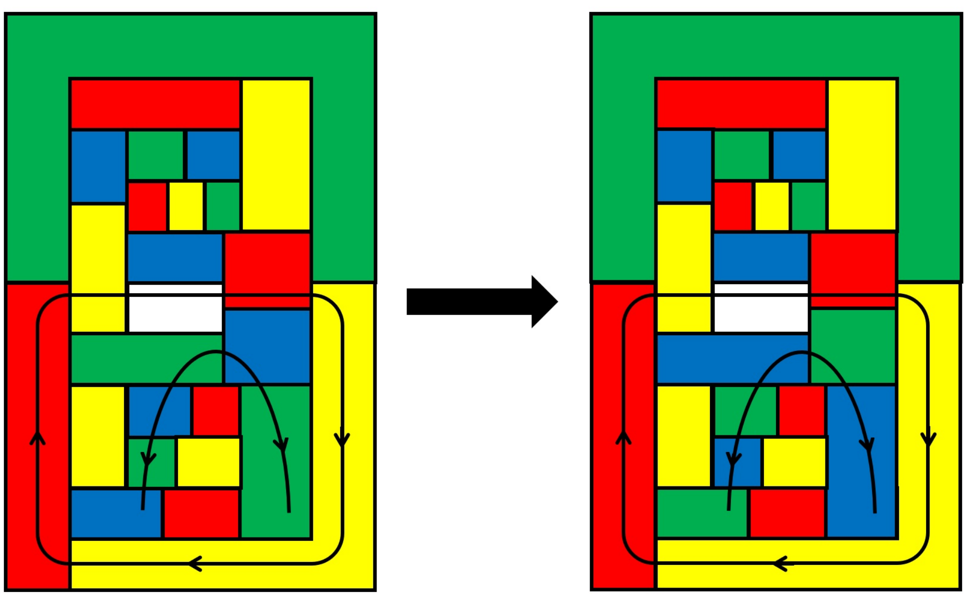
Percy John Heawood (September 8, 1861–January 24, 1955)

Image from the MacTutor History of Mathematics Archive biography of Heawood (accessed 12/23/2022)

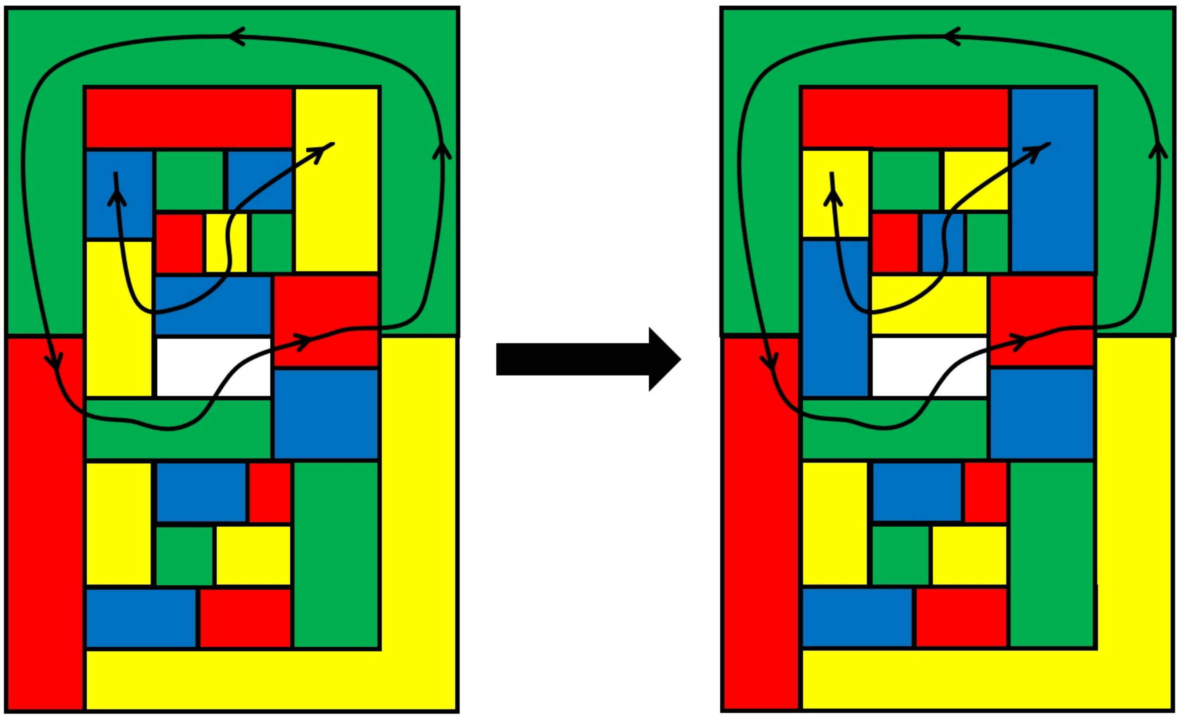
Below is the map of Heawood's paper and a more schematic (but colorful) version. Notice that the pentagon of Heawood's Figure 18 has two red neighbors. In the colorized version of this we have interchanged the red and blue countries to be consistent with the explanation of Kempe's "proof" above. We still discuss red-green chains and red-yellow chains in the context of linkage.



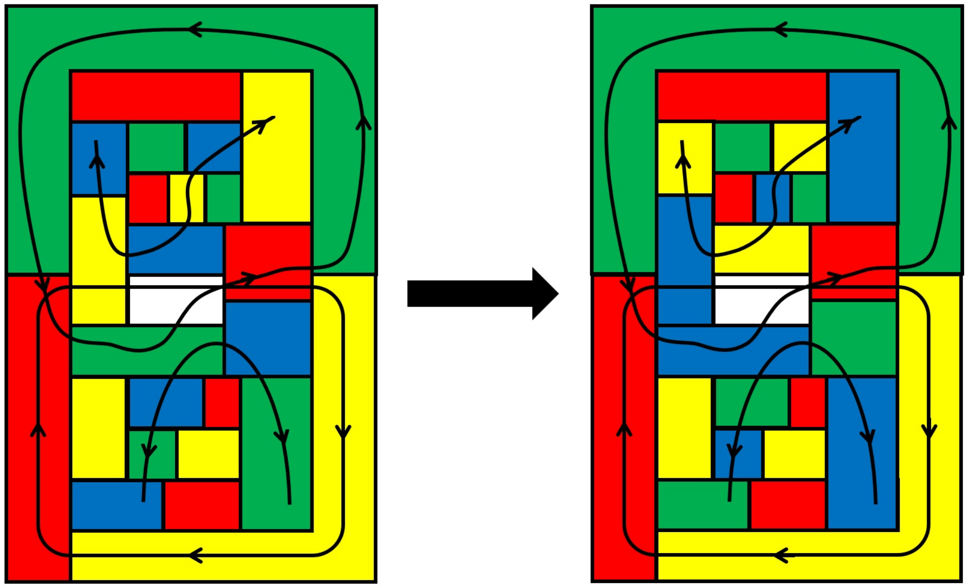
In our schematic diagram, the pentagon is white. Applying Kempe's approach, we find a red-yellow chain starting at the red country bordering the pentagon and see that it links to the yellow country bordering the pentagon. So we start with the blue country bordering the pentagon that is inside this link, and find the corresponding blue-green chain. We then interchange the blue and green countries in this chain with no color conflicts.



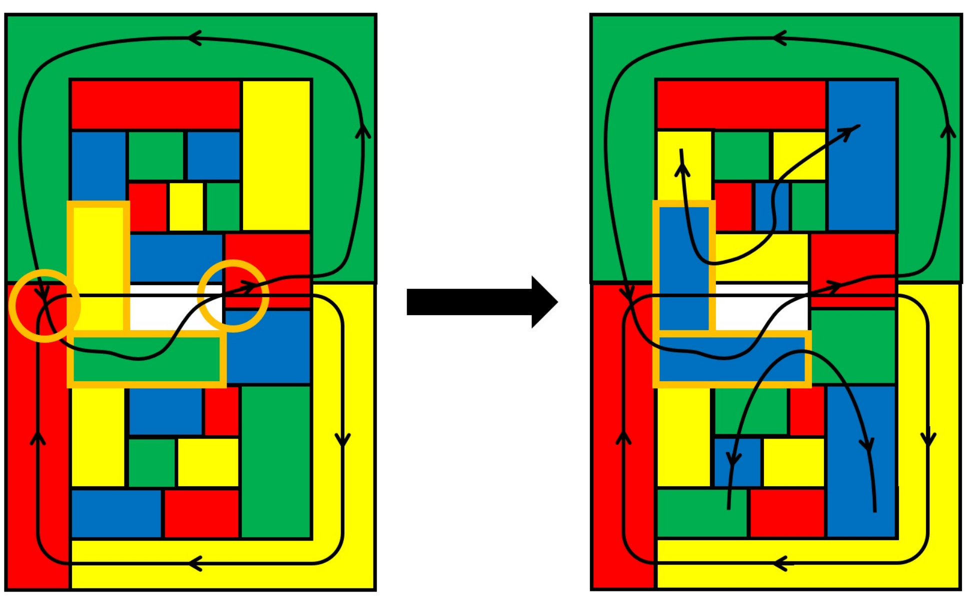
Next, we find the red-green chain starting at the red country bordering the pentagon and see that it links around to the green country bordering the pentagon. So we start with the blue country ordering the pentagon that is inside this link, and find the corresponding blue-yellow chain. We then interchange the blue and yellow countries in this chain with no color conflicts.



Following Kempe's technique, we now combine the blue-yellow interchange and the blue-green interchange. However, this leads to a color conflict since it results in two blue countries sharing a border. Since Kempe's construction fails for this particular map, it is not in general correct and Kempe's proof is not valid.



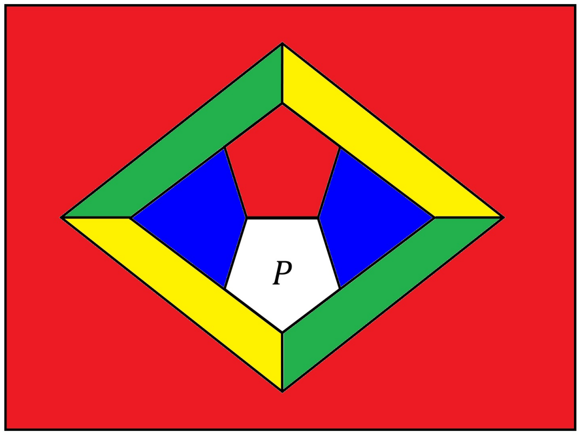
The next diagram illustrates the real problem. Since Kempe considers red-green chains that link together and red-yellow chains that link together, there is the risk that red countries could be shared between these two chains and that they could even cross each other. This destroys the argument about the separation of the recolored countries in the blue-yellow and blue-green chains "inside" the links.



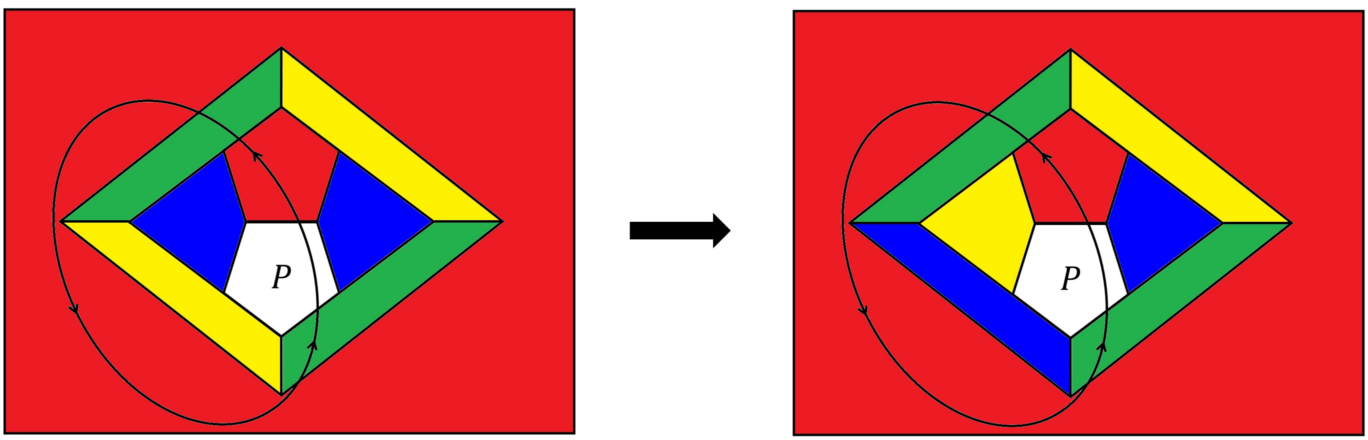
In this diagram (left), the orange circles indicate where the red-green link and the red-yellow link cross each other. Kempe's procedure then recolors the yellow countries inside the red-green link (including the troubled yellow country with an orange boundary) as blue, and recolors the green countries inside the red-yellow link (including the troubled green country with an orange boundary) as blue. Notice that these two troubled countries are *not* separated by the red-green and red-yellow links! This is how Kempe's procedure fails and results in the two neighboring blue countries (right).

Note. We make a passing comment that in Wilson's presentation of Heawood's argument (see pages 120–123), Wilson considers changing colors of countries *outside* of a link instead of inside of a link as we did above; this simply results in some permutation of colors of various two-color chains. A color conflict still results. Presumably, Wilson is following the argument made in Heawood's 1890 paper.

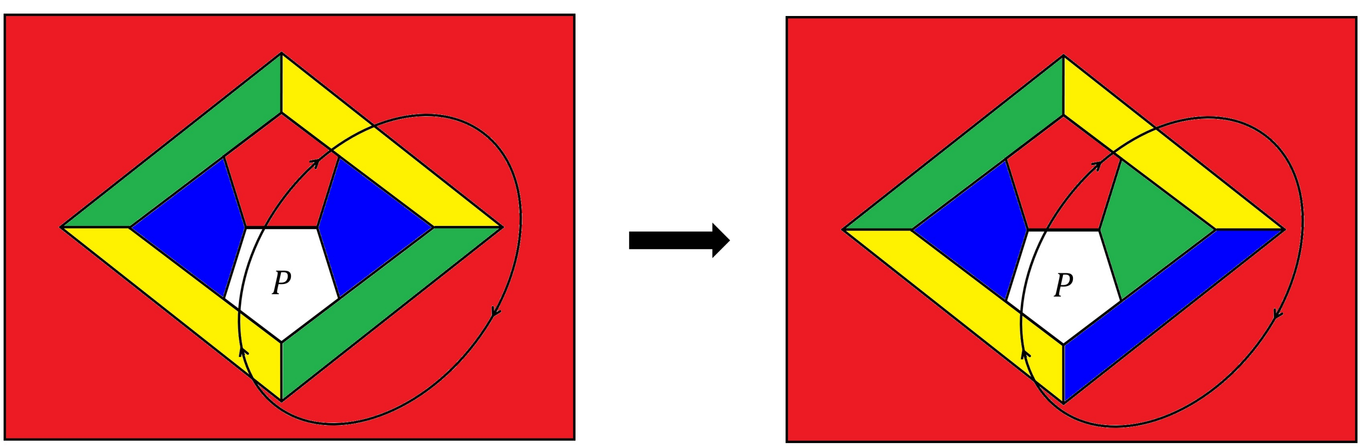
Note. A smaller example illustrating the failure of Kempe's approach under the case of a pentagon and the existence of two links, is given in J. A. Bondy and U. S. R. Murty's graduate level *Graph Theory* as Exercise 15.2.2. The intended solution to the exercise is more formal than what we now present. This example takes a bit more of a modern approach by considering a map defined on the entire plane (or, equivalently, on a sphere), so that we have an outer, unbounded country. Consider the following map.



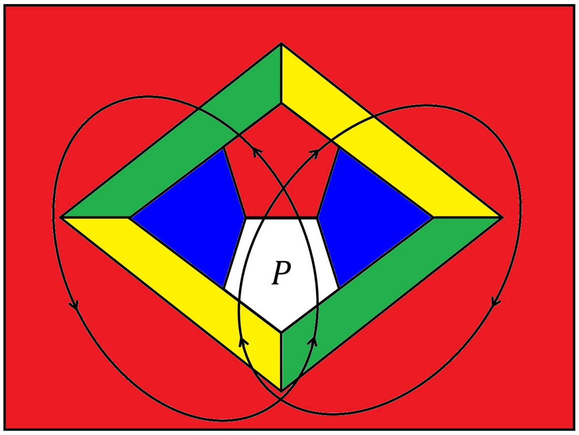
The red-green link and corresponding blue-yellow interchange are as follows.

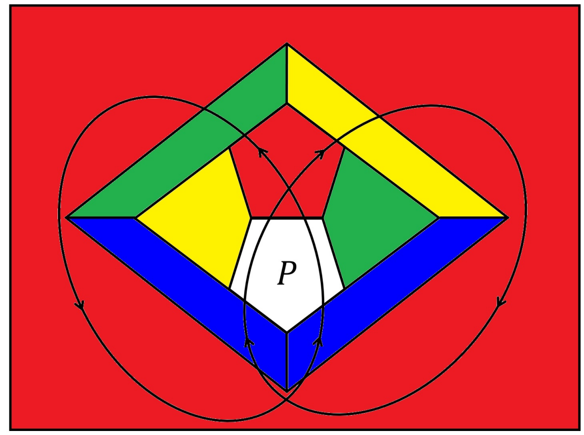


The red-yellow link and corresponding blue-green interchange are as follows.



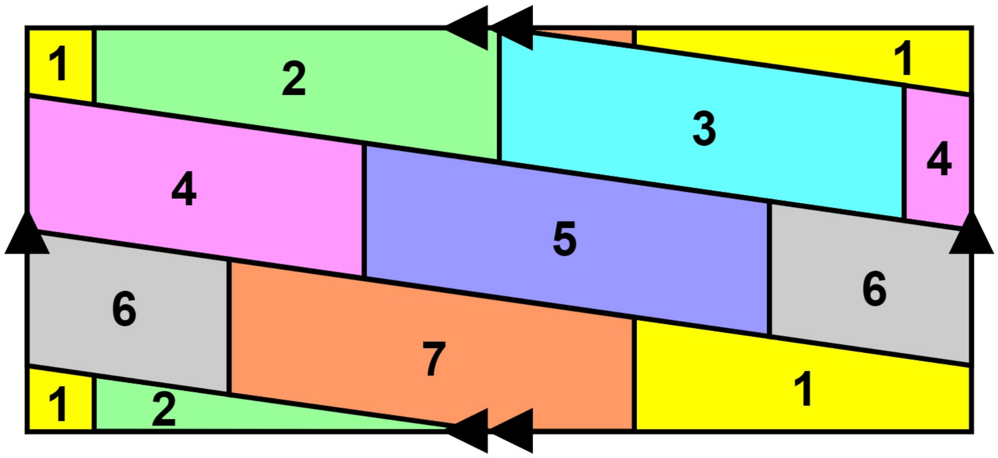
Combining the two color interchanges reveals the problem.





Again, the red-green link and the red-yellow link cross (at the two red countries). As with Heawood's example, this results in two countries which are not separated by the links and the color interchanges produce two neighboring blue countries.

Note. We conclude Heawood's story with the observation that his 1890 paper was not all negative. It included a proof of the Five Color Theorem. That is, he proved that every map can be colored using at most five colors. His proof is inductive on the number of countries (or, equivalently, on the number of vertices in the dual graph). This is covered in Graph Theory 1 (MATH 5340) in Section 11.2. The Five-Colour Theorem; see Theorem 11.6 where the proof given is basically that of Heawood. Heawood also gave an example of a map with seven countries on the surface of a torus in which each country shares a boundary with every other country. The following is an image of such a configuration. The image is from the Wikipedia Four Color Theorem website (accessed 12/26/2022).



Using Euler's Formula for a torus, one can prove the "Only Six Neighbors Theorem for the Torus." That is, every map on a torus has at least one country with six or fewer neighbors. From this, one can prove that every map on the torus can be colored with seven colors. Wilson gives details on these torus results on his pages 133–137.

Revised: 4/25/2023