Section 1.1. Graphs

Note. We present Godsil and Royle's quick review of graph theory terminology. A detailed account of these ideas can be found in any undergraduate graph theory text. I recommend J.A. Bondy and U.S.R. Murty's *Graph Theory with Applications* (1976, NY: North Holland) which is available from Web.Archive. This book is aimed at upper level undergraduates and beginning graduate students (that is, it is appropriate for the cross listed "Introduction to Graph Theory" class [MATH 4347/5347]). A related source is Bondy and Murty's *Graph Theory*, Graduate Texts in Mathematics #244 (2008, Springer). I have notes posted online for this second source. If you have some background in graph theory, then you can probably skip the first two sections of this chapter. However, there is value in familiarizing yourself with notation used by Godsil and Royle.

Definition. A *graph* X consists of a *vertex set* $V(X)$ and an *edge set* $E(X)$, where an edge is an unordered pair of distinct vertices of X. For $x, y \in V(X)$ we denote the edge consisting of the unordered pair x, y as xy. For $xy \in E(X)$ we say that x and y are *adjacent* or that y is a *neighbor* of x, denoted $x \sim y$. A vertex is *incident* with an edge it it is one of the two vertices of the edge. Two graphs X and Y are *equal* if $V(X) = V(Y)$ and $E(X) = E(Y)$.

Note. Of course we are not as interested in equal graphs as we are in isomorphic graphs. As in other settings, an isomorphism is a bijection between two mathematical structures which preserves structure. The structure of a graph is adjacency.

Definition. Graphs X and Y are *isomorphic* if there is a bijection $\varphi : V(X) \to V(Y)$ such that $x \sim y$ in X if and only if $\varphi(x) \sim \varphi(y)$. We say that φ is an *isomorphism* from X to Y. We denote X is isomorphic to Y as $X \cong Y$.

Note. As usual, we we draw graphs by representing the vertices as points (in this book represented as little circles) and the edges as continuous curves connecting the vertices to which the edge is incident. Such a drawing is not meant to imply any additional structure (such as a location of the vertices), but only reflects the vertices and edges of the graph (and if we do not label the edges or vertices then these give a representation of a graph "up to isomorphism"). Godsil and Royle give tow different drawings of the same graph on five vertices in Figure 1.1.

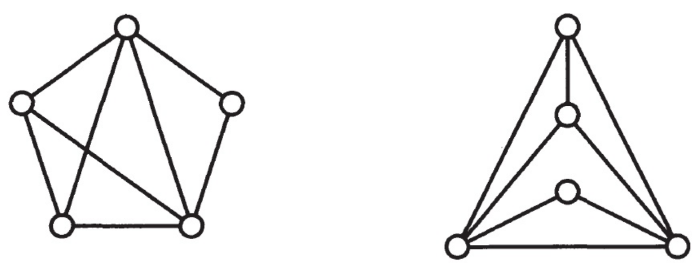

Figure 1.1. Two graphs on five vertices

Definition. A graph is *complete* if every pair of vertices are adjacent, and the complete graph on n vertices is denoted K_n. A graph with no edges (and at least one vertex) is *empty*. The graph with no vertices (and hence no edges) is the *null graph*. A graph X for which there are no edges of the form xx and no edges are repeated (implicit in the fact that we have defined the edge *set*) is *simple*; we may consider at some point multigraphs which have edge multisets (which allow for repetition of edges).

Definition. A *directed graph* (or *digraph*) X consists of a *vertex set* $V(X)$ and an *arc set* $A(X)$, where an *arc*, or *directed edge*, is an ordered pair of distinct vertices.

Note. We can also draw a directed graph by representing the vertices as points and the arcs as directed continuous curves where the direction is reflected by an arrow. See Figure 1.2.

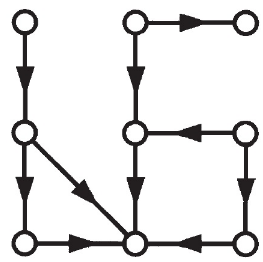

Figure 1.2. A directed graph

Note. We adopt the following standards in these notes. We explicitly say if we are considering directed graphs (as opposed to graphs). **When we use the term "graph" we mean a simple graph, finite graph** (that is, a simple graph with a finite number of vertices).

Revised: 7/19/2020