Section 1.3. Automorphisms

Note. We define an automorphism of a graph and consider some properties of a graph preserved under an automorphism.

Definition. An isomorphism from a graph X to itself is an *automorphism* of X. The set of all automorphisms of graph X form a group under function composition called the *automorphism group* of X, denoted $\text{Aut}(X)$. The group of all permutations of the vertex set $V(X)$ is the *symmetric group* $\text{Sym}(V) = \text{Sym}(n)$ where $|V| = n$ (so $\text{Sym}(V)$ is isomorphic to the symmetry group S_n, and $\text{Aut}(X) < \text{Sym}(V)$ where "<" denotes the relation of subgroup).

Note. The fact that $\text{Aut}(X)$ actually is a group is shown in Exercise 1.2.9 of Bondy and Murty's *Graph Theory*, Graduate Texts in Mathematics #244 (2008, Springer).

Note. For $v \in V(X)$ and $g \in \text{Sym}(V)$, we denote the image of v under g as v^g. If $g \in \text{Aut}(X)$ and Y is a subgraph of X, then Y^g denotes the graph with vertex set $V(Y^g) = \{x^g \mid x \in V(Y)\}$ and edge set $E(Y^g) = \{x^g y^g \mid xy \in E(Y)\}$. Then Y^g is isomorphic to Y (under isomorphism g) and Y^g is also a subgraph of X.

Definition. The *valency* (or more traditionally **degree**) of a vertex x is the number of neighbors of x. The *maximum and minimum valency of a graph* X are the maximum and minimum values of the valencies of any vertex of X (traditionally denoted $\delta(X)$ and $\Delta(X)$).

Lemma 1.3.1. If x is a vertex of the graph X and g is an automorphism of X, then the vertex $y = x^g$ has the same valency as x. (That is, an automorphism preserves degrees of vertices.)

Definition. A graph in which every vertex has equal valency k is *regular of valency* k or k*-regular*. A 3-regular graph is *cubic*, and a 4-regular graph is *quartic*.

Definition. The *distance* $d_X(x, y)$ (also denoted $d(x, y)$ if the graph is clear from the context) between two vertices x and y in a graph X is the length of the shortest path from x to y.

Lemma 1.3.2. If x and y are vertices of X and $g \in \mathrm{Aut}(X)$, then $d(x, y) = d(x^g, y^g)$. (That is, an automorphism preserves distances between vertices.)

Note. A proof of Lemma 1.3.2 is to be given in Exercise 1.3 (we number exercises as Chapter.Number since exercises are only given at the ends of chapters).

Definition. The *complement* $\overline{X}$ of a graph X has the same vertex set as X, where vertices x and y are adjacent in $\overline{X}$ if and only if they are not adjacent in X. See Figure 1.5.

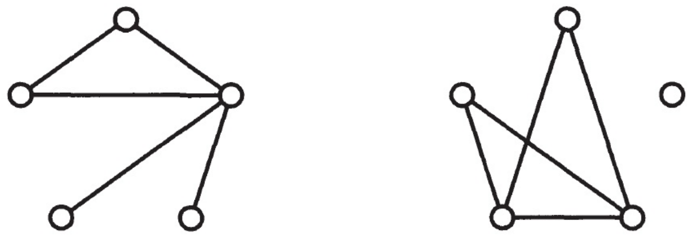

Figure 1.5. A graph and its complement

Lemma 1.3.3. The automorphism group of a graph is equal to the automorphism group of its complement.

Note. A proof of Lemma 1.3.3 is to be given in Exercise 1.2.

Note. We could similarly define an automorphism g of a directed graph as a bijection from the vertex set onto itself which preserves the arc set (so (x, y) is an arc of the directed graph if and only if (x^g, y^g) is also an arc).

Revised: 7/19/2020