Section 1.4. Homomorphisms

Note. We define a homomorphism between graphs and between directed graphs. We also relate vertex colourings to homomorphisms.

Definition. Let X and Y be graphs. A mapping $f : V(X) \mapsto V(Y)$ is a *homomorphism* if $f(x)$ and $f(y)$ are adjacent in Y whenever x and y are adjacent in X.

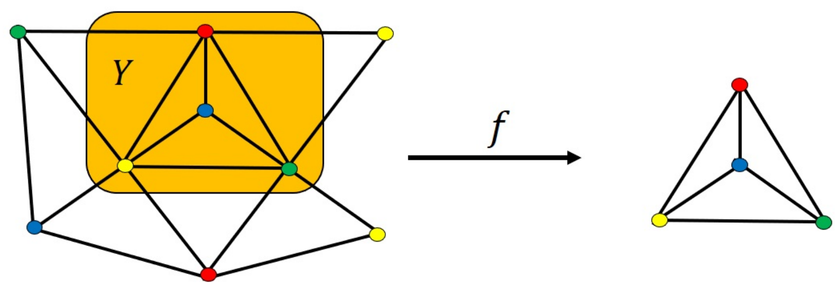
Note. Since we do not allow loops in a graph, then for $x \sim y$ we have $f(x) \neq f(y)$. So homomorphism f may not be one-to-one (injective) but it cannot map adjacent vertices in X to the same vertex of Y. Any isomorphism is a homomorphism, but of course the converse need not hold. We next relate vertex colourings of a graph to homomorphisms.

Definition. A *proper colouring* of a graph X is a map from $V(X)$ into some finite set of colours such that no two adjacent vertices are assigned the same colour. If X can be properly coloured with a set of k colours, then we say that X is *properly k-colourable*. The least value of k for which X can be properly k-coloured is the *chromatic number* of X, denoted $\chi(X)$. The set of all vertices of a given colour is a *colour class* (and is an independent set since no two such vertices can be adjacent).

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Definition. A *retraction* is a homomorphism f from a graph X to Y of itself such that the restriction $f|Y = f|_Y$ of f to $V(Y)$ is the identity map. If there is a retraction from X to a subgraph Y, then we say that Y is a *retract* of X.

Note. Suppose graph X has a clique Y (i.e., a complete subgraph) of size $k = \chi(X)$. Then we can define a homomorphism f from X to Y such that $f(x) = y$ if and only if x and y are the same colour. If x_1 and x_2 are different colours in X then $f(x_1) \neq f(x_2)$. Since $f(x_1)$ is adjacent to all $f(x_2)$ for $f(x_1) \neq f(x_2)$, then f satisfies the definition of homomorphism. Also, f maps any vertex of Y to itself, so $f|Y$ is the identity and hence f is a retraction:



Definition. Let X and Y be directed graphs. A mapping $f : V(X) \mapsto V(Y)$ is a *homomorphism* if $(f(x), f(y)) \in A(X)$ whenever $(x, y) \in A(Y)$.

Definition. A homomorphism from a graph X to itself is an *endomorphism* and the set of all endomorphisms of X form the *endomorphism monoid* of X.

Note. Since an automorphism of a graph X is also an endomorphism (because an isomorphism is also a homomorphism), then the automorphism groups $\mathrm{Aut}(X)$ is contained in the endomorphism monoid of X (as a submonoid). Recall that a monoid is a binary algebraic structure where the binary operation is associative and has an identity (see my online notes for Modern Algebra 1 [MATH 5410] on Section I.1. Semigroups, Monoids, and Groups).

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