Section 1.5. Circulant Graphs

Note. We define a circulant graph and a circulant directed graph. Godsil and Royle state on page 8 that these graphs are "an important class of graphs that will provide useful examples in later sections."

Note. Consider a graph X with $V(X) = \{0, 1, \ldots, n-1\}$ and

$$E(X) = \{i\,j \mid (j - i) \equiv \pm 1 \pmod{n}\}.$$

This is the graph C_n, the cycle on n vertices. If g is the element of the symmetry group $S_n = \mathrm{Sym}(n)$ which maps i to $(i + 1)(\mathrm{mod}\ n)$, then $g \in \mathrm{Aut}(C_n)$ (as is easily verified). Notice that g is an elementary rotation of the cycle C_n; notice that $g^n = \iota$, the identity permutation. Since $g \in \mathrm{Aut}(C_n)$ then the cyclic group $R = \{g^m \mid 0 \le m \le n-1\}$ of order n is a subgroup of $\mathrm{Aut}(C_n)$. If h is the element of the symmetry group $S_n = \mathrm{Sym}(n)$ which maps i to $-i(\mathrm{mod}\ n)$, then $h \in \mathrm{Aut}(C_n)$ (as is easily verified). Notice that h produces a mirror image of C_n (a "reflection" about an axis of the cycle that passes radially through vertex 0); notice that $h^2 = \iota$. So $\mathrm{Aut}(C_n)$ includes all products of powers of g and h. These two permutations generate the dihedral group, D_n (see my online notes for Modern Algebra 1 [MATH 5410] on Section I.6. Symmetric, Alternating, and Dihedral Groups; see Theorem 6.13). In fact, in Exercise 1.2.10 of Bondy and Murty's s *Graph Theory*, Graduate Texts in Mathematics #244 (2008, Springer), it is to be shown that $\mathrm{Aut}(C_n) = D_n$.

Definition. Let $\mathbb{Z}_n$ denote the additive group of integers modulo n (so $\mathbb{Z}_n$ is a cyclic group of order n). Let $C \subseteq \mathbb{Z}_n \backslash \{0\}$. Define the directed graph $X = X(\mathbb{Z}_n, C)$ to have vertex set $V(X) = \mathbb{Z}_n$ and arc set $A(X) = \{(i, j) \mid (j - i) \pmod{n} \in C\}$. The graph $X(\mathbb{Z}_n, C)$ is a *circulant directed graph of order* n and C is the *connection set*.

Definition. Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. Let $C \subseteq \mathbb{Z}_n \setminus \{0\}$ satisfy $i \in C$ implies $-i(\text{mod } n) \in C$ (that is, C is closed under additive inverses). Define the graph $X = X(\mathbb{Z}_n, C)$ to have vertex set $V(X) = \mathbb{Z}_n$ and edge set $E(X) = \{i\, j \mid (j - i) \pmod{n} \in C\}$. The graph $X(\mathbb{Z}_n, C)$ is a *circulant graph of order* n and C is the *connection set*.

Note. Since $(j - i) \pmod{n} \in C$ if and only if $((j + 1) - (i + 1)) \,(\text{mid } n) \in C$, then every circulant graph and directed graph admits the permutation that sends i to $i + 1(\text{mod } n)$ is an automorphism. Hence, the automorphism group of a circulant graph or directed graph has a cyclic subgroup of order n. If C is closed under additive inverses (which is the case for a circulant [undirected] graph) then the permutation that sends i to $(-i) \pmod{n}$ is an automorphism and then the automorphism group has a dihedral subgroup of order $2n$.

Note. The cycle C_n is a circulant graph of order n with connection set $S = \{-1, 1\}$. So, in this sense, circulant graphs are generalizations of cycles. It is shown in Exercises 1.3.18 and 1.3.19 of Bondy and Murtys *Graph Theory*, Graduate Texts in

Mathematics #244 (2008, Springer), that circulant (undirected) graphs are special cases of "Cayley graphs." A Cayley graph is a graph built from a group G where the vertex set of the graph is the set of elements of group G, with vertices x and y adjacent if and only if $x - y \in S$ where S is the connection set; a circulant graph is then a Cayley graph where the group is $G = \mathbb{Z}_n$.

Note. The complete graph is a circulant graph with connection set $C = \mathbb{Z}_n$. See Figure 1.7 for another example. We expect a circulant graph to have lots of symmetries (and so large automorphism groups).

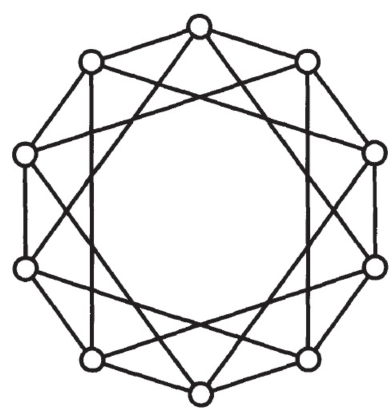

Figure 1.7. The circulant $X(\mathbb{Z}_{10}, \{-1, 1, -3, 3\})$