1.5. Circulant Graphs

Section 1.5. Circulant Graphs

Note. We define a circulant graph and a circulant directed graph. Godsil and Royle state on page 8 that these graphs are “an important class of graphs that will provide useful examples in later sections.”

Note. Consider a graph $X$ with $V(X) = \{0, 1, \ldots, n - 1\}$ and

$$E(X) = \{i j \mid (j - i) \equiv \pm 1 \pmod{n}\}.$$  

This is the graph $C_n$, the cycle on $n$ vertices. If $g$ is the element of the symmetry group $S_n = \text{Sym}(n)$ which maps $i$ to $(i + 1)(\text{mod} \ n)$, then $g \in \text{Aut}(C_n)$ (as is easily verified). Notice that $g$ is an elementary rotation of the cycle $C_n$; notice that $g^n = \iota$, the identity permutation. Since $g \in \text{Aut}(C_n)$ then the cyclic group $R = \{g^m \mid 0 \leq m \leq n-1\}$ of order $n$ is a subgroup of $\text{Aut}(C_n)$. If $h$ is the element of the symmetry group $S_n = \text{Sym}(n)$ which maps $i$ to $-i(\text{mod} \ n)$, then $h \in \text{Aut}(C_n)$ (as is easily verified). Notice that $h$ produces a mirror image of $C_n$ (a “reflection” about an axis of the cycle that passes radially through vertex 0); notice that $h^2 = \iota$. So $\text{Aut}(C_n)$ includes all products of powers of $g$ and $h$. These two permutations generate the dihedral group, $D_n$ (see my online notes for Modern Algebra 1 [MATH 5410] on Section I.6. Symmetric, Alternating, and Dihedral Groups; see Theorem 6.13). In fact, in Exercise 1.2.10 of Bondy and Murty’s s Graph Theory, Graduate Texts in Mathematics #244 (2008, Springer), it is to be shown that $\text{Aut}(C_n) = D_n$. 
**Definition.** Let $\mathbb{Z}_n$ denote the additive group of integers modulo $n$ (so $\mathbb{Z}_n$ is a cyclic group of order $n$). Let $C \subseteq \mathbb{Z}_n \setminus \{0\}$. Define the directed graph $X = X(\mathbb{Z}_n, C)$ to have vertex set $V(X) = \mathbb{Z}_n$ and arc set $A(X) = \{(i, j) \mid (j - i) \mod n \in C\}$. The graph $X(\mathbb{Z}_n, C)$ is a circulant directed graph of order $n$ and $C$ is the connection set.

**Definition.** Let $\mathbb{Z}_n$ denote the additive group of integers modulo $n$. Let $C \subseteq \mathbb{Z}_n \setminus \{0\}$ satisfy $i \in C$ implies $-i \mod n \in C$ (that is, $C$ is closed under additive inverses). Define the graph $X = X(\mathbb{Z}_n, C)$ to have vertex set $V(X) = \mathbb{Z}_n$ and edge set $E(X) = \{i \cdot j \mid (j - i) \mod n \in C\}$. The graph $X(\mathbb{Z}_n, C)$ is a circulant graph of order $n$ and $C$ is the connection set.

**Note.** Since $(j - i) \mod n \in C$ if and only if $((j + 1) - (i + 1)) \mod n \in C$, then every circulant graph and directed graph admits the permutation that sends $i$ to $i + 1 \mod n$ is an automorphism. Hence, the automorphism group of a circulant graph or directed graph has a cyclic subgroup of order $n$. If $C$ is closed under additive inverses (which is the case for a circulant [undirected] graph) then the permutation that sends $i$ to $-i \mod n$ is an automorphism and then the automorphism group has a dihedral subgroup of order $2n$.

**Note.** The cycle $C_n$ is a circulant graph of order $n$ with connection set $S = \{-1, 1\}$. So, in this sense, circulant graphs are generalizations of cycles. It is shown in Exercises 1.3.18 and 1.3.19 of Bondy and Murty’s *Graph Theory*, Graduate Texts in
Mathematics #244 (2008, Springer), that circulant (undirected) graphs are special cases of “Cayley graphs.” A Cayley graph is a graph built from a group $G$ where the vertex set of the graph is the set of elements of group $G$, with vertices $x$ and $y$ adjacent if and only if $x - y \in S$ where $S$ is the connection set; a circulant graph is then a Cayley graph where the group is $G = \mathbb{Z}_n$.

**Note.** The complete graph is a circulant graph with connection set $C = \mathbb{Z}_n$. See Figure 1.7 for another example. We expect a circulant graph to have lots of symmetries (and so large automorphism groups).

![Figure 1.7. The circulant $X(\mathbb{Z}_{10}, \{-1, 1, -3, 3\})$](image)

*Revised: 7/20/2020*