Section 1.6. Johnson Graphs

Note. We introduce a very general class of graphs, denoted $J(v, k, i)$ that will "recur throughout the book" (according to Godsil and Royle, see page 9). This class of graphs will be useful in couching some combinatorial problems in the graph theoretic terms.

Definition. Let v, k, and i be fixed positive integers, with $v \geq k \geq i$. Let Ω be a fixed set of size v, and define graph $J(v, k, i)$ as follows. The vertices of $J(v, k, i)$ are the subsets of Ω with size k, where two subsets are adjacent if their intersection has size i.

Note 1.6.A. Since $|\Omega| = v$ then Ω has $\binom{v}{k}$ subsets of size k so that graph $J(v, k, i)$ has $\binom{v}{k}$ vertices. Now the number of subsets of size i of a set of size k is $\binom{k}{i}$ (so a given vertex "has" $\binom{k}{i}$ subsets). For one of these subsets of size i, we look for the number of other subsets of size k which contain the same i elements (and hence the number of other vertices that are adjacent to the given vertex because of the intersection of the two vertices being the given set of size i). Such a subset of size k would contain the same i elements and any $k - i$ other elements of ω (of which there are $v - k$ such elements in Ω). So the number of such sets (i.e., vertices) is $v - k$ choose $k - i$, $\binom{v-k}{k-i}$. So the total valency (i.e., "degree") of a vertex is the product of the number of subsets of a set of size k times the number of other subsets of size k which contain the same i elements. That is, the valency of each vertex is $\binom{k}{i}\binom{v-k}{k-i}$ and the graph is regular with each vertex of this valency.

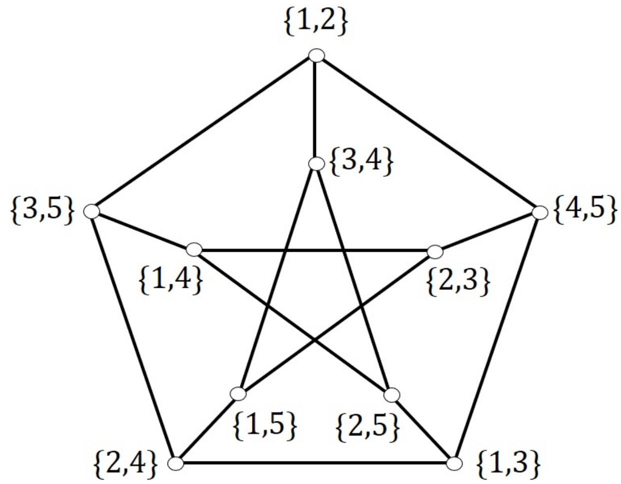
Lemma 1.6.1. If $v \geq k \geq i$, then $J(v,k,i) \cong J(v, v-k, v-2k+i)$.

Note. If $v/2 < k$ (i.e., $v < 2k$), then $v - k < k$ and $v \geq v - k \geq v - k - (k - i) = v-2k+i$. By Lemma 1.6.1, with $v \geq k \geq i$ we have $J(v,k,i) \cong J(v, v-k, v-2k+i)$, where the vertices of $J(v,k,i)$ are subsets of Ω of size k, and the vertices of $J(v, v-k, v-2k+1)$ are subsets of Ω of size $v - k > v - v/2 = v/2 > k$. So for $v < 2k$ we can replace a $J(v,k,i)$ with the isomorphic structure $J(v, v-k, v-2k+i)$. Hence when considering $J(v,k,i)$, we can without loss of generality assume $v > 2k$.

Definition. For $v \geq 2k$ the graphs $J(v,k,k-1)$ are the *Johnson graphs*. The graphs $J(v,k,0)$ are the *Kneser graphs*.

Note. We'll consider Kneser graphs in detail in Chapter 7 (and their eigenvalues in Section 9.4). Exercise 1.3.8 of Bondy and Murtys *Graph Theory*, Graduate Texts in Mathematics #244 (2008, Springer), explores a couple of special cases of Kneser graphs.

Note. The Petersen graph is an example of a Kneser graph, namely it is $J(5,2,0)$. Let $\omega = \{1,2,3,4,5\}$ so that the vertices of $J(5,2,0)$ are the subsets of Ω of size 2: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$. Since $i = 0$, then two vertices are adjacent if and only if they are disjoint:



Note. If g is a permutation of set Ω and $S \subseteq \Omega$, then define the set S^g as $S^g = \{s^g \mid s \in S\}$.

Lemma 1.6.2. If $v \geq k \geq i$, then $\mathrm{Aut}(J(v,k,i))$ contains a subgroup isomorphic to $\mathrm{Sym}(v)$.

Note. An automorphism of $J(v,k,i)$ is a permutation on a set of size $\binom{v}{k}$. So when $k \neq 1$ or $k \neq v-1$ (notice $\binom{v}{1} = \binom{v}{v-1} = v$ and $\binom{v}{k} \neq v$ otherwise) we have that $\mathrm{Aut}(J(v,k,i))$ is not *equal* to $\mathrm{Sym}(v)$, though it could still be *isomorphic* to $\mathrm{Sym}(v)$ (and, as Godsil and Royle comment on page 10, the automorphism group "usually" is isomorphic to $\mathrm{Sym}(v)$, though they do not elaborate).

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