Section 1.7. Line Graphs

Note. Line graphs are briefly introduced in Graph Theory 1 (MATH 5340) in Section 1.3. Graphs Arising from Other Structures. Here, we go into more detail and prove a few results concering line graphs.

Definition. The *line graph* of a graph X is the graph $L(X)$ with the edges of X as its vertices, and where two edges of X are adjacent in $L(X)$ if and only if they are incident in X.

Note. Figure 1.9 gives a graph X (in grey) and its line graph $L(X)$ (in black). The vertices of $L(X)$ are in a position reflecting their correspondence with the edges of X.

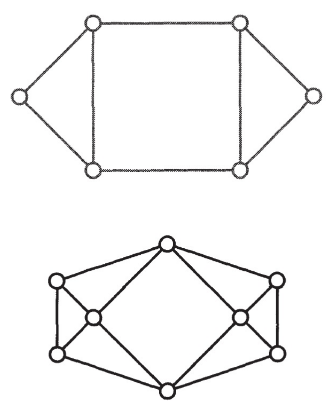


Figure 1.9. A graph and its line graph

As more general examples, we observe that the star graph $K_{1,n}$ (in which every one of the n edges is incident to every other edge) has the compete graph K_n as it line graph (in which every one of the n vertices is adjacent to every other vertex). The path with n vertices, P_n, has vertex set $\{1, 2, \ldots, n\}$ where i is adjacent to $i+1$ for

$1 \leq i \leq n-1$; it line graph is the path P_{n-1}. The cycle C_n is (isomorphic to) its own line graph.

Lemma 1.7.1. If X is regular with valency (i.e., degree) k, then $L(X)$ is regular with valency $2k-2$.

Proof. Since each vertex of of X is of valency k, then every edge of X is incident with $k-1$ edges at each of its ends. Hence, in $L(X)$ each vertex of of valency $2(k-1) = 2k-2$, as claimed. ∎

Theorem 1.7.2. A nonempty graph is a line graph if and only if its edge set can be partitioned into a set of cliques with the property that any vertex lies in at most two cliques.

Note. The edge disjoint cliques of the line graph of Figure 1.9 (lower), as described in Theorem 1.7.2, are: the left most edge (a K_2), the right most edge (a K_2), the upper left triangle (a K_3), the upper right triangle (a K_3), the lower left triangle (a K_3), the lower right (a K_3). Notice this is six cliques and the graph yielding this line graph (Figure 1.9, upper) has six vertices.

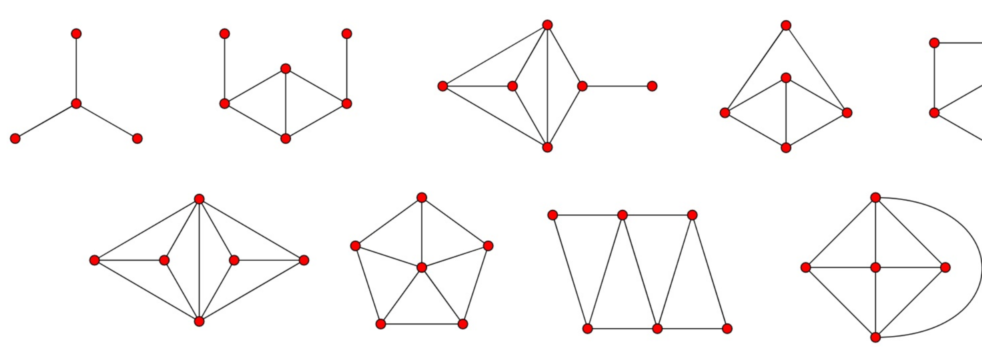
Note. If $X \cong Y$, then $L(X) \cong L(Y)$. But the converse does not hold, as seen by the fact that K_3 and $K_{1,3}$ have the same line graph, namely $K_3 = C_3$. It was shown in 1932 that this is the only pair of connected nonisomorphic graphs with the property that their line graphs are the same in H. Whitney, "Congruent Graphs and the Connectivity of Graphs," *American Journal of Mathematics*, **54**(1), 150–168 (1932) (available on JSTOR, accessed 9/10/2022). Here, we prove a special case of this more general result, as follows.

Lemma 1.7.3. Suppose that X and Y are graphs with minimum valency four. Then $X \cong Y$ if and only if $L(X) \cong L(Y)$.

Note. Another characterization of line graphs appears in: L.W. Beineke, *Derived Graphs of Digraphs*, in H. Sachs, H.J. Voss,and H.J. Walter (eds.), Beiträge zur Graphentheorie, 17–33 (1968). It is the following:

Theorem 1.7.4. A graph X is a line graph if and only if each induced subgraph of X on at most six vertices is a line graph.

Note. By Theorem 1.7.4, the set of graphs X such that (a) X is not a line graph, and (b) every proper induced subgraph of X is a line graph, is finite. In fact, there are nine "forbidden" graphs in this set. They are given as follows on the Wikipedia Line Graph webpage (accessed 9/10/2022):



Definition. A bipartite graph is *semiregular* if it has a 2-colouring such that all vertices with the same colour have the same valency.

Note. Easy examples of semiregular graphs are complete bipartite graphs $K_{m,n}$, where the vertices in a partite set are assigned the same colour. The next lemma addresses the line graphs of connected regular graphs.

Lemma 1.7.5. If the line graph of a connected graph X is regular, then X is regular or bipartite and semiregular.

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