Section 16.1. Knots and Their Projections

Note. In this section we define knot and link, and give diagrams and projections (or "shadows") to represent them. We define the equivalence of knots in terms of piecewise-linear bijections and consider mirror-images of knots. No proofs are given in this section.

Definition. A *knot* is a piecewise-linear closed curve in $\mathbb{R}^3$. A *link* is a collection of pairwise disjoint knots; the knots constituting a link are its *components*.

Note. Sometimes a knot is defined in terms of the image of an interval $[a, b]$ under a continuous one to one function $f : [a, b] \to \mathbb{R}^3$. See, for example, my online notes for Introduction to Knot Theory on 2.1. Wild Knots and Unknottings. However, this approach allows in certain structures which do not agree with the desired idea of a physical knot. For example, a such a definition would include knots with infinite numbers of crossings. This is avoided by the piecewise-linear-based definition (such a knot is sometimes called a closed polygonal curve, as in Livingston's *Knot Theory*). As is standard, we draw knots and links using smooth curves, which we want to think of as made up of very small line segments.

Note/Definition. Figure 16.1 shows how we will illustrate graphs and links. Such pictures are called *link diagrams* and, in a 2-dimensional diagram, they show how a 3-dimensional knot is embedded in 3-space. These diagrams include the "under-and-over" information necessary to reflect the third dimension. We require that the diagrams satisfy: (a) at most two points on the link correspond to a given point in its diagram (neither of these points can be the end of a segment; they are where we have "crossings"), and (b) only finitely many points in the diagram correspond to more than one point on the link (so no segments of two linear curves overlap).

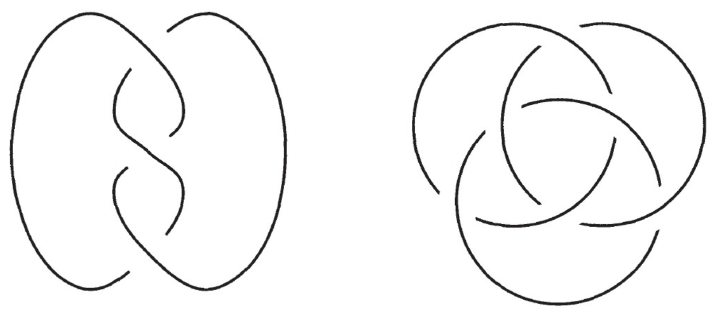

Figure 16.1. A knot diagram and a link diagram

Note/Definition. We can convert a diagram of a link into a drawing of a planar graph by replacing each crossing with a vertex and loosing the under-and-over information. Notice that this results in a 4-regular graph. This is called the *shadow* of the link diagram. These are called "projections" in Livingston's *Knot Theory*, but they are not treated as graphs in Livingston. Of course, neither diagrams nor shadows are unique for a given knot/link. Figure 16.2 gives a diagram and a shadow.

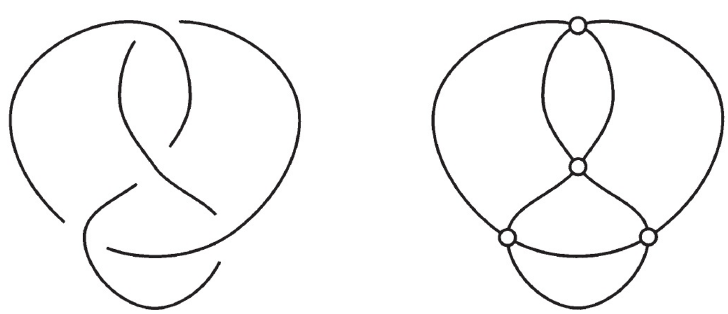

Figure 16.2. A knot diagram and its shadow

Definition. A *homeomorphism* is a piecewise-linear bijection. Two links L_1 and L_2 are *equivalent* if there is an orientation-preserving homeomorphism φ from $\mathbb{R}^3 \cup \{\infty\}$ to itself that maps L_1 onto L_2.

Note. We need to elaborate some here. Normally in a topology class, we take the definition of homeomorphism as a continuous bijection with a continuous inverse (see my online notes for Introduction to Topology [MATH 4357/5357] on Section 18. Continuous Functions). Here, we are consider piecewise-linear bijections, which are certainly continuous. So our definition is different from what you might see elsewhere, but it is sufficient for our purposes here. We can consider $\mathbb{R}^3 \cup \{\infty\}$ as a "one-point compactification of $\mathbb{R}^3$ (see the Introduction to Topology notes on Section 29. Local Compactness; notice Theorem 29.1 and Example 4). In this way, the 3-sphere S^3 is "the same as" (i.e., homeomorphic to) $\mathbb{R}^3 \cup \{\infty\}$. Similarly, $\mathbb{R} \cup \{\infty\}$ (the extended real line) is homeomorphic to S^1 and $\mathbb{R}^2 \cup \{\infty\}$ is homeomorphic to S^2 (this homeomorphism is often illustrated using stereographic projection and the Riemann sphere; see my online notes for Complex Analysis 1 [MATH 5510] on I.6. The Extended Plane and Its Spherical Representation).

Definition. A knot is the *unknot* if it is isotopic to a circle lying in a plane.

Note. Figure 16.3 gives two diagrams of the unknot. Notice that even for this elementary case, it is not obvious that the diagram on the left represents the unknot. We need a procedure to convert one diagram to the other in such a way that we can see the equivalence of the knots represented by the two diagrams.

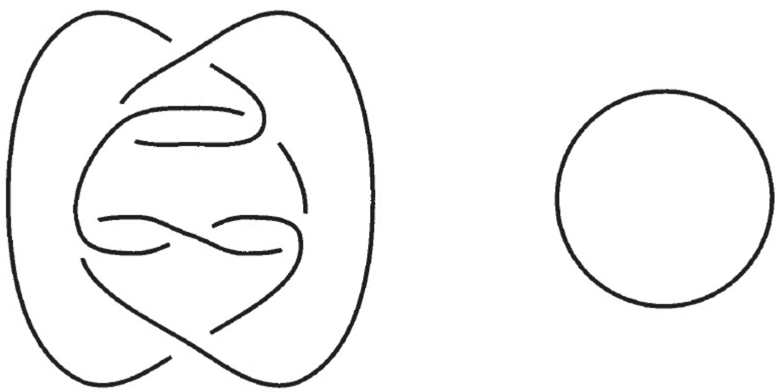

Figure 16.3. Two diagrams of the unknot.

Definition. Given a link L we can form it mirror image L' be reflecting L in a plane through the origin.

Note. We can find a mirror image of a knot in $\mathbb{R}^3$ by replacing every point (x, y, z) by, say, $(-x, y, z)$. This is a mirror image that results from reflecting the knot about the yz-plane. Links L and L' may or may not be equivalent.

Definition. A link that is equivalent to its mirror image is *achiral*.

Note. Figure 16.4 gives the trefoil knot and its mirror image. The trefoil is not achiral. We will prove this in Section 16.2, "Signed Plane Graphs."

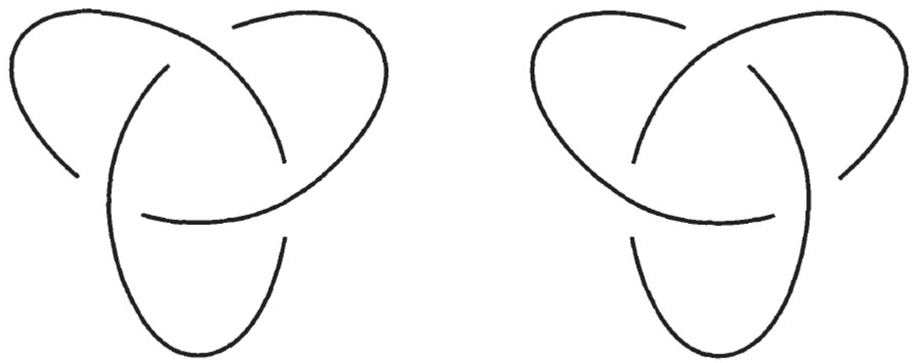

Figure 16.4. The right-handed and left-handed trefoil

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