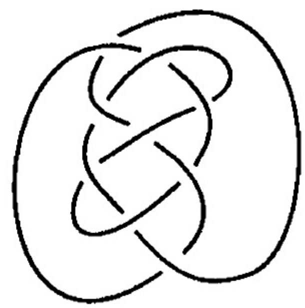
Chapter 16. Knots

Note. Godsil and Royle very concisely state (page 373): "A *knot* is a closed curve of finite length in $\mathbb{R}^3$ that does not intersect itself." A more detailed (though still rather informal) discussion of knots can be found in Charles Livingston's *Knot Theory*, The Carus Mathematical Monographs, Volume 24, (MAA, 1993). See my online notes for Introduction to Knot Theory (not a formal ETSU class) for more information. A very detailed (and rigorous) discussion of knots is given in W. B. Raymond Lickorish's *An Introduction Knot Theory*, Graduate Texts in Mathematics #175, (NY: Springer-Verlag, 1997). See my online notes for Graduate Knot Theory (also not a formal ETSU class) for more information.

Note. The equivalence of knots is a fundamental problem in knot theory. Two knots are equivalent if (informally) one can be "deformed" into the other without one strand of a knot passing through another strand in the deformation. If we claim that two knots are equivalent, then we could simply show the steps that deforms one into the other (using, say, the Reidemeister moves which are introduced in Section 16.4). However, if we claim that two knots are not equivalent then this approach is not useful. The standard way to show two knots are not equivalent is to find an "invariant" of the knots which are differ. We will present graphs in terms of diagrams (defined in Section 16.1) and a given knot may have several diagrams. An "invariant" of a knot which is based on a diagram would be something that is the same for all diagrams of the knot.

Note. Much of knot theory is topological in nature, involving associating a topological property with a knot. In these notes, we concentrate more on combinatorial properties of knots. An early combinatorial invariant of knots is the Alexander polynomial (see Section 3.5. The Alexander Polynomial in the Livingston reference and Chapter 6. The Alexander Polynomial in the Lickorish reference). Two non-equivalent knots can have the same Alexander polynomial. For example, the polynomials 9_{28} and 9_{29} given below both have the Alexander polynomial $t^6 - 5t^5 + 12t^4 - 15t^3 + 12t^2 - 5t + 1$:

$\mathbf{9_{28}}$



$\mathbf{9_{29}}$

From Appendix A "Knot Table" of Livingston's *Knot Theory* (MAA, 1993)

In 1985 Vaughan Jones introduced a new polynomial, now called the Jones polynomial, which is able to distinguish knots with the same Alexander polynomial (see page 215 of the Livingston reference); though the computation of the Jones polynomial can be very complicated (and recursive). Jones' presented his work in "A Polynomial Invariant for Knots via von Neumann Algebras," *Bulletin of the American Mathematical Society*, **12**(1), 103–111 (1985), which can be viewed online at the Bulletin of the A.M.S. webpage. We present the Kaufmann Bracket in Section 16.6 and use it in Section 16.7 to introduce the Jones polynomial.

Note. A brief history of knot theory can be found in my online notes for Introduction to Knot Theory on Chapter 1. A Century of Knot Theory.

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