Chapter 1. Introduction

Note. In this brief chapter we review some basic definitions and terminology. We use the material of this book as a supplement to J.A. Bondy and U.S.R. Murty's *Graph Theory*, Graduate Texts in Mathematics 244 (Springer, 2008). These notes do not include all definitions from Chapter 1, but only those that differ somewhat from Bondy and Murty's and those of central importance to our focus. Here, the focus is on topological graph theory. We consider, in particular, trees, bipartite graphs, blocks, and connectivity.

Section 1.1. Basic Definitions

Note. Bondy and Murty define a graph G as a structure consisting of a vertex set $V(G)$ an edge set $E(G)$, and an incidence function ψ_G which maps $E(G)$ into the set of unordered (not necessarily distinct) pairs of vertices. In this way, when they use the term "graph" they are allowing parallel edges and loops (see my online notes for Graph Theory 1 [MATH 5340] on 1.1. Graphs and Their Representations). They define a "simple graph" as one with no parallel edges and no loops. Mohar and Thomassen reserve the term "graph" for a simple graph, and for a graph with parallel edges or loops they use the term "multigraph." This allows them to avoid the use of an incidence function in their definition of "graph" and they then refer to edges as unordered pairs of vertices.

Definition. A (simple) *graph* G is a pair of sets $V(G)$ and $E(G)$ where $V(G) \neq \varnothing$ and $E(G)$ is a set of 2-element subsets of $V(G)$. The elements of $V(G)$ are *vertices* and the elements of $E(G)$ are *edges*. For edge $e = \{x, y\} \in E(G)$, vertices x and y are the *ends* (or *endvertices*) of e. The cardinality of the vertex set $|V(G)|$ is the *order* of graph G. Vertices u and v are *adjacent* if there is an edge of G having u and v as its ends, and in this case u and v are *neighbors*.

Note. Unless stated otherwise, **we assume all graphs are of finite order**, $|V(G)| < \infty$. For $e = \{u, v\} \in E(G)$, we commonly write $e = uv = vu$ (the edge being unambiguously determined by its ends in a simple graph).

Definition. A *multigraph* G is an ordered triple $G = (V(G), E(G), \partial)$ where $V(G)$ is a (finite) nonempty set of *vertices*, $E(G)$ is a (finite) set of *edges*, and ∂ is a function that assigns to each edge $e \in E(G)$ a (unordered) pair of (not necessarily distinct) vertices (namely, the *ends* of e). Distinct edges which have the same ends are *parallel edges* or *multiple edges*. An edge where the two ends are the same vertex is *loop*. The *degree* of a vertex $v \in V(G)$ is the number of edges that have v as an end, counting each loop twice.

Note. The function ∂ here plays the same role as the incidence function ψ_G of Bondy and Murty. If we allow *multisets* (that is, collections of elements where an element can be repeated in the collection; the number of times an element is repeated is the *multiplicity* of the element) then we can define multigraphs similar to the way we define (simple) graphs. We just take the vertex set $V(G)$ to be nonempty and let the edge multiset $E(G)$ have as its elements the 2-element multisets of vertices. Adding a subscript of "m to denote a multiset (for example $A = \{a, a, b, c, c, c\}_m$) we have for the multigraph below (modified from Bondy and Murty's Figure 1.1(a)) that $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uu, uv, vw, wx, wx, xy, xu\}_m$.

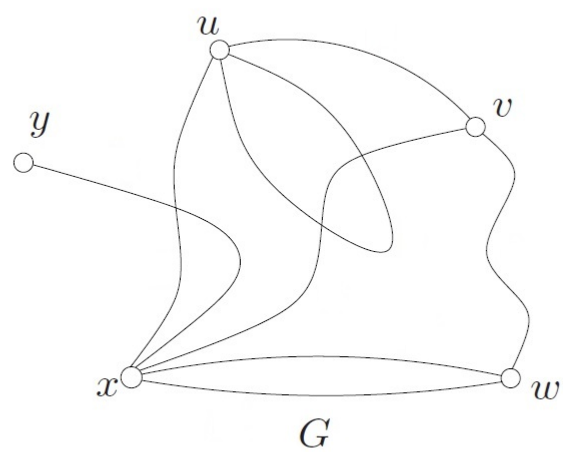


Figure 1.1(a) (from Bondy and Murty, modified)

Note. Mohar and Thomassen comment (see page 3):

> "Most of the results that we shall derive for graphs carry over to multigraphs in a natural and obvious way. Because of slightly easier presentation and since the more general framework of multigraphs does not yield any stronger results, we decided to **limit our presentation to [simple] graphs** whenever possible. [emphasis added]"

Definition. An *isomorphism* of graphs G and H is a one to one (injective) mapping ψ of $V(G)$ onto $V(H)$ such that adjacent pairs of vertices of G are mapped to adjacent vertices of H, and nonadjacent pairs of vertices have nonadjacent images. If there is such an isomorphism, then G and G are *isomorphic*, which we denote $G = H$.

Note. The "structure" of a graph is the adjacency. So an isomorphism between two graphs is a one to one and onto mapping (i.e., a bijection) that preserves structure.

Definition. A graph G is *connected* if two of its vertices are connected by a path in g. A *connected component* of a graph G is a maximal connected subgraph of G.

Definition. A subgraph C of G isomorphic to C_n (a cycle of length n) is called an *n-cycle in G*. An edge e of G joining two nonconsecutive vertices of C is a *chord* of C. If C has no chords, it is called *chordless* or an *induced cylce* in G. A complete subgraph of G is a *clique* in G.

Note. We now give some definitions concerning creating a new graph from a given G by making changes to the vertex and/or edge set of G. This will be useful, for example, in classifying planar graphs.

Definition. For $X \subseteq V(G)$, we have that $G - X$ denotes the subgraph of G obtained by deleting from G the vertices in X and all edges incident with them. If v is a vertex in G then we denote $G - \{v\}$ as $G - v$, called a *vertex-deleted subgraph* of G. If $A \subseteq E(G)$ then $G - A$ is the subgraph of G obtained by removing from G the edges in A. If e is an edge in G then we denote $G - \{e\}$ as $G - e$. If $u, v \in V(G)$ are nonadjacent vertices of G, then $G - uv$ denotes the graph obtained from G by adding edge uv (so that $E(G + uv) = E(G) \cup \{uv\}$).

Definition. Let $e = uv$ be an edge of graph G and denote by G/e the graph (or multigraph) obtained from G by removing the edge e and identifying its ends u and v as a single new vertex. This operation is called *edge contraction*. If $A \subseteq E(G)$ then G/A denotes the graph (or multigraph) obtained by successively contracting all edges in A.

Note. It is left as Exercise 1.1.1 to show that for $A = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$, the order in which we perform the successive edge contraction does not affect G/A. The notation "G/A" is inspired by the notation of cosets of a group from abstract algebra where bunches of group elements are identified (see my online notes for Introduction to Modern Algebra [MATH 4127/5127] on II.10. Cosets and the Theorem of Lagrange). We might be tempted to read "G/A" as "G modulo A" here, but instead we read it as "G with edge contraction set A." Figure 1.2 represents a graph G on the left with $A \subseteq E(G)$ as the bold faced edges, and the edge contracted multigraph G/A on the right.

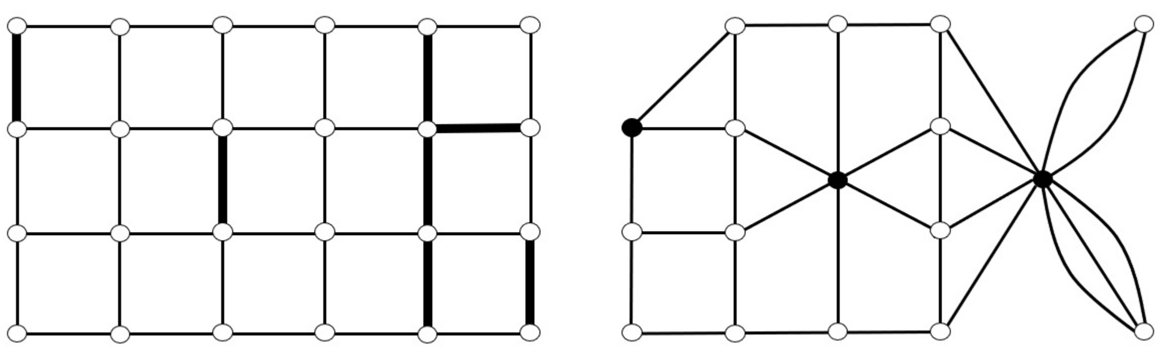
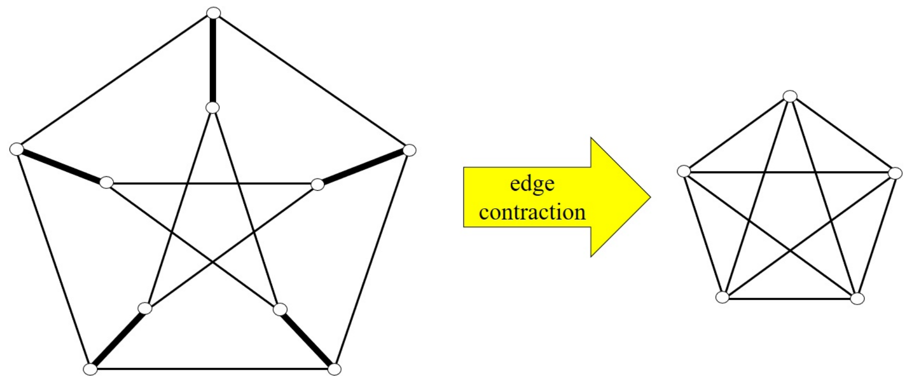


Figure 1.2. The bold faced edges on the right are contracted, resulting in the graph on the left where the new identified vertices are in black. Based on Mohar and Thomassen's Figure 1.2.

Definition. A graph H is a *minor* of graph G if H is isomorphic to a graph obtained from a subgraph G' of G by contracting a set of edges $A \subset E(G')$.

Example 1.1.A. The Petersen graph has K_5 as a minor. In the notation of Figure 1.2 above, we contract the bold faced edges of the Petersen graph as follows:



Note. A minor of G is of the form G'/A where G' is a subgraph of G so that $G' = (G-B)-C$ for some $B \subseteq E(G)$ and some $C \subseteq V(G)$ where the vertices in C are (some of the) isolated vertices in $G - B$. So every minor o fG can be obtained from G by successively contracting edges (to get G/A) and deleting edges (to get $(G/A)-B$) and deleting isolated vertices (to get $((G/A)-B)-C$). In Exercise 1.1.2 it is to be shown for $A, B \subseteq E(H)$, where $A \cap B = \varnothing$, that $(H/A)-B = (H-B)/A$. So $(G/A(-B-(G-B)/A$ and the order in which the edge contractions and edge deletions are performed do not affect the minor graph. Of course, in performing these operations, once a vertex appears as isolated then it can (if desired) be removed, or it can be remoced later (if desired). That is, the (1) edge contractions, (2) edge deletions, and (3) removal of isolated vertices, can be performed in any order. Therefore, any minor of G is determined (up to a set of isolated vertices) by two disjoint sets $A, B \subseteq E(G)$ where A is the set of edges to be contracted and B is the set of edges to be removed. Bondy and Murty define a minor as a graph resulting from a sequence of vertex and edge deletions in G (see 10.5. Kuratowskis Theorem).

Note. The book describes *vertex splitting* as the inverse operation of edge contraction. Splitting vertex v in simple graph G is the replacement of v by adjacent vertices v' and v'' and the replacement of each edge $e = uv$ incident to v with either edge $v'u$ or edge $v''u$ (but not both) to create (simple graph G'. Bondy and Murty describe this for multigraphs in 2.3 Modifying Graphs. See Figure 1.3 for an example. Notice that the contraction of edge $v'v''$ in G' results in the original graph G.

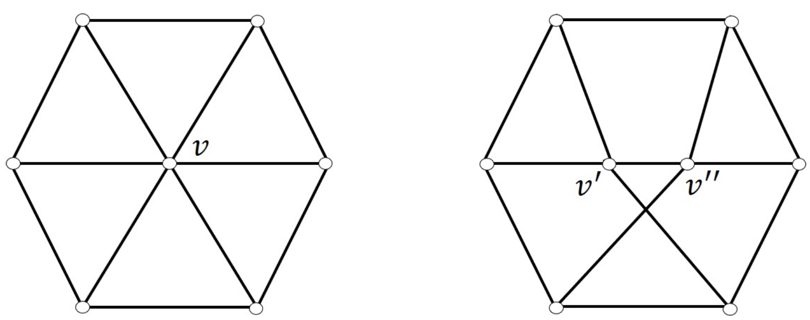


Figure 1.3. Splitting of vertex v. Based on Mohar and Thomassen's Figure 1.3.

Definition. A graph H is a *subdivision* of graph G if $H = G$ or if H can be obtained from G by replacing some edges of G by paths such that each of these paths has only its endpoints in common with G.

Note. Bondy and Murty define subdividing an edge by replacing an edge uv with two new edges uw and wv. Then they define a subdivision of a graph as a graph obtained by a sequence of edge subdivisions.

Definition. Graphs G and H are *homeomorphic* if there is a third graph K which is isomorphic to a subdivision of G and isomorphic to a subdivision of H.

Note. In a topological setting, a homeomorphic is a bijection between two topological spaces which is continuous with a continuous inverse. See my online notes for Introduction to Topology (MATH 4357/5357) on 2.18. Continuous Functions. A homeomorphism between topological spaces is as an isomorphism between other mathematical structures. Homeomorphic topological spaces have the "same shape." This is the idea behind the definition of a graph homeomorphism. Figure 1.4 gives two homeomorphic graphs, neither of which is a subdivision of the other.

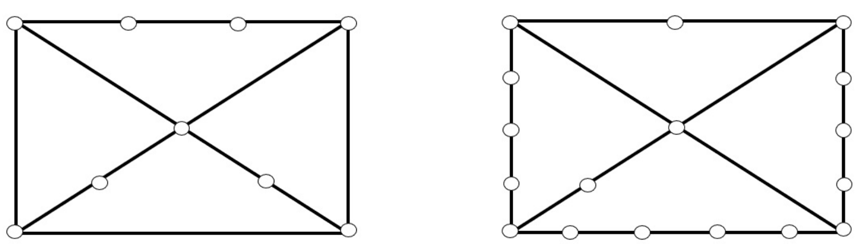


Figure 1.4. Homeomorphic graphs neither of which is a subdivision of the other. Based on Mohar and Thomassen's Figure 1.4.

Note. Bridges in a graph (in fact, in a multigraph) are explored in detail in Bondy and Murty's 10.4. Bridges. We now give a definition equivalent to their's.

Definition. Let H be a subgraph of G. An H*-bridge* in G (or a "bridge of H in G") is a subgraph of G is either an edge not in H but with both ends in H (called by Bondy and Murty a *trivial* H-bridge; the trivial H-bridge includes its ends in H), or a connected component of $G - V(H)$ together with all edges (and their ends in H) which have one end in the component and the other end in H. Let B be an H-bridge. Vertices of $B \cap H$ are *vertices of attachment* (or simply *attachments*) and each edge of B incident with a vertex of attachment is a *foot* of B.

Note. The edge sets of H-bridges partition the set $E(G) \setminus E(H)$. The figure below gives an example of the four bridges associated with a cycle C is a graph.

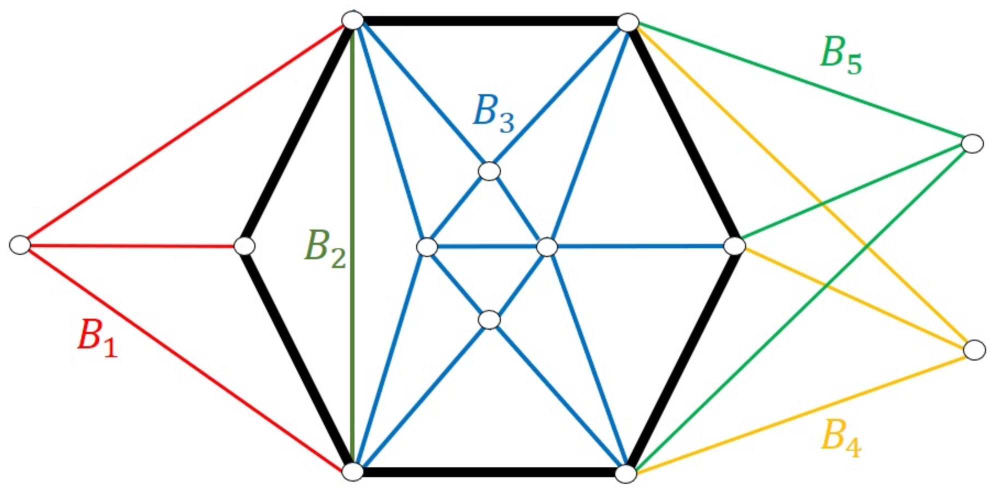


Figure. A cycle C (in bold), along with the C-bridges B_1, B_2, B_3, B_4, and B_5. Based on Mohar and Thomassen's Figures 1.5 and 1.6.

Note. For C a cycle in a graph G, the "interaction" of the C-bridges in G relate to whether G is planar or not. For example, Theorem 10.26 of Bondy and Murty (in 10.4. Bridges) states that for plane graph G containing cycle C, the inner bridges of X avoid one another and the outer bridges of C avoid one another. "Inner" and "outer" here is used in the sense of the Jordan Curve Theorem and are based on a particular given embedding of G in the plane. The term "avoid" as used here is the negation of the term "overlap," defined next.

Definition. Let C be a cycle in a graph G. Two C-bridges B_1 and B_2 *overlap* if either

(i) B_1 and B_2 have these vertices of attachment in common, or

(ii) C contains distinct vertices a, b, c, d (in this cyclic order) such that a and c are vertices of attachment of B_1 and b and d are vertices of attachment of B_2.

If (ii) holds then B_1 and B_2 *skew-overlap* (or are "skew" in the terminology of Bondy and Murty).

Note. In the figure above, the following pairs of bridges overlap: B_3 and B_4, B_3 and B_5, and B_4 and B_5. The pairs B_4 and B_5 do not skew-overlap, while the other two pairs do skew-overlap. Notice that each edge of B_1, B_2, B_4, and B_5 is a foot. Bridge B_3 has nine edges which are "feet."

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