# **Section 1.2.** Trees and Bipartite Graphs

**Note.** We consider two common classes of graphs and state a few results about them.

**Definition.** A *tree* is a connected graph that contains no cycles (i.e., an acyclic connected graph). A graph that contains no cycles (and is possibly disconnected) is a *forest*. The vertices of a tree of degree 1 are *endvertices* of the tree. For a given graph, a spanning subgraph which is a tree is a *spanning tree*.

**Note.** By Bondy and Murty's Theorem 4.6, a graph is connected if and only if it has a spanning tree

**Note.** The next result gives some classifications of trees.

**Proposition 1.2.1.** Let $T$ be a graph of order $n$. Then the following are equivalent.

**(i)** $T$ is a tree.

**(ii)** $T$ is connected and has $n-1$ edges.

**(iii)** $T$ contains no cycles and has $n-1$ edges.

**(iv)** $T$ is connected but every edge-deletion results in a disconnected graph.

**(v)** $T$ contains no cycles but every edge addition results in a graph with a cycle.

**(vi)** Any two vertices in $T$ are connected by exactly one path.

**Note.** The next result relates subgraphs of a given graph $G$ which are trees to a minor of $G$.

**Proposition 1.2.2.** A graph $H$ with vertices $v_1, v_2, \ldots, v_k$ is a minor of graph $G$ if and only if $G$ contains pairwise disjoint trees $T_1, T_2, \ldots, T_k$ such that for $1 \le i < j \le k$ there is an edge between a vertex of $T_i$ and a vertex of $T_j$ whenever $v_i$ and $v_j$ are adjacent in $H$.

**Note.** Mohar and Thomassen give an example in their Figure 1.9 which illustrates the idea of the proof of Proposition 1.2.2 in the case that the $3 \times 4$ grid graph $G$ contains a minor of $K_5$ minus an edge. Notice that the minor results from deleting all the vertices of $G$ which are not in any of the trees, and contracting all the edges in the trees (thus forming the vertices of the minor). We now formalize this in a proof of Proposition 1.2.2.

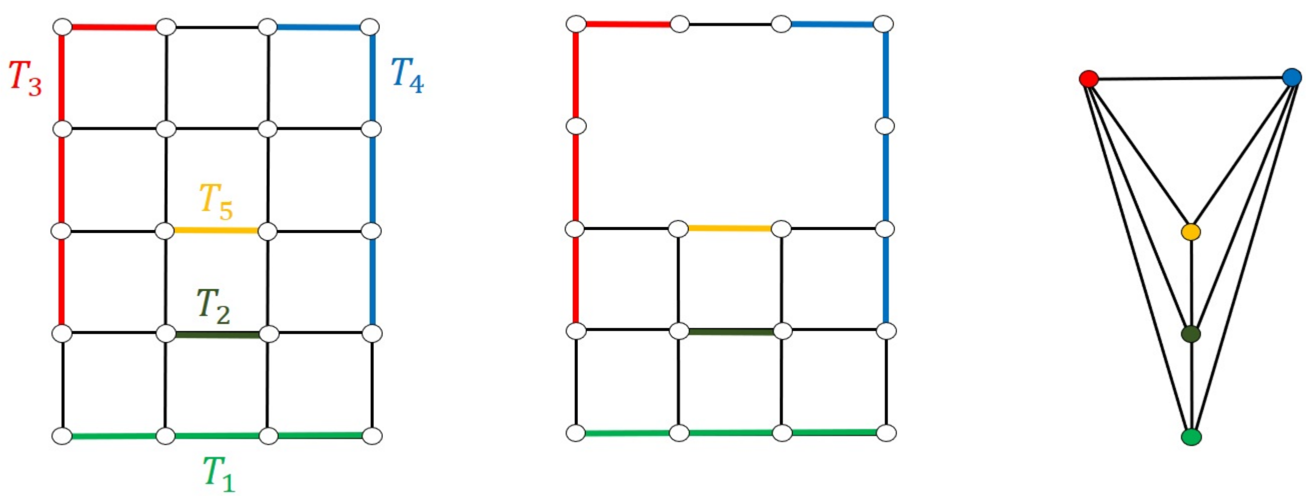


**Figure 1.9.** A $3 \times 4$ grid graph $G$ (left), $G$ with vertices in no subtrees deleted (center), and the graph that then results form contractions of the edges in each subtree.

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