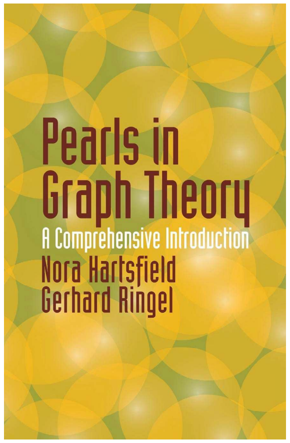
Introduction to Graph Theory

Chapter 1. Basic Graph Theory

1.1. Graphs and Degrees of Vertices—Proofs of Theorems



Theorem 1.1.1

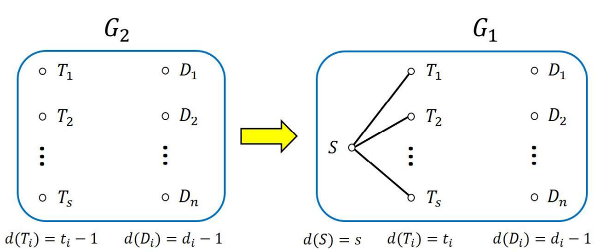
Theorem 1.1.1. Let $v_1, v_2, \ldots, v_p$ be the vertices of a graph G, and let $d_1, d_2, \ldots, d_p$ be the degrees of the vertices, respectively. Let q be the number of edges of G. Then

$$d_1 + d_2 + \cdots + d_p = \sum_{i=1}^{p} d_i = 2q.$$

Proof. By definition, an edge e of G is incident to two distinct vertices, namely its endpoints, say v_i and v_j. So any given edge e contributes (an amount of 1) to two of the degrees, say d_i and d_j. Hence each edge of G accounts for an amount of 2 in the sum $d_1 + d_2 + \cdots + d_p$. That is, the sum is twice the number of edges, $d_1 + d_2 + \cdots + d_p = 2q$, as claimed. □

Theorem 1.1.2

Theorem 1.1.2. (Havel, Hakimi) Consider the following two sequences and assume sequence (1) is in descending order.
(1) $s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_n$
(2) $t_1 - 1, t_2 - 1, \ldots t_s - 1, d_1, d_2, \ldots, d_n$.
The sequence (1) is graphic if and only if sequence (2) is graphic.
Proof. First assume that sequence (2) is graphic. Then, by definition of "graphic," there is a graph $G_2 = (V_2, E_2)$ with degree sequence (2). We construct graph G_1 from graph G_2 by adding a single vertex S and adding s edges incident to S as follows:



Theorem 1.1.2 (continued 1)

Proof (continued). Symbolically, construct graph $G_1 = (V_1, E_1)$ where $V_1 = V_2 \cup \{S\}$ (where S is a new vertex not in V_2) and E_1 is the set of edges consisting of all edges in E_2 along with s edges where each of these s edges has S as one endpoint and the other endpoint is one of the vertices of G_2 of degree $t_1 - 1, t_2 - 1, \ldots, t_s - 1$ (and each these s vertices of G_2 appear as an endpoint of exactly one of the new edges). In terms of the symbols introduced in the figure above,
$E_1 = E_2 \cup \{ST_i \mid i \in \{1, 2, \ldots, s\}\}$. Then in graph G_1, vertex S is of degree s, each vertex of T_i has degree t_i, and each vertex D_i has degree d_i. So graph G_1 has the sequence (1) as its degree sequence and so (1) is graphic, as claimed.

Theorem 1.1.2 (continued 2)

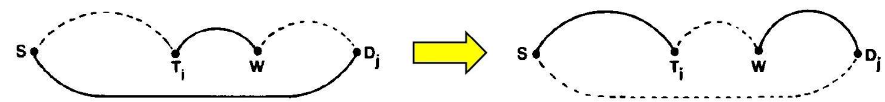
Theorem 1.1.2. (Havel, Hakimi) Consider the following two sequences and assume sequence (1) is in descending order.
(1) $s, t_1, t_2, \dots, t_s, d_1, d_2, \dots, d_n$
(2) $t_1 - 1, t_2 - 1, \dots t_s - 1, d_1, d_2, \dots, d_n$.
The sequence (1) is graphic if and only if sequence (2) is graphic.

Proof (continued). Now suppose the sequence (1) is graphic. Then, by definition of "graphic," there is a graph H with degree sequence (1). Denote the vertices of H of degree t_i and T_i, the vertices of degree d_i as D_i, and the vertex of degree s as S. We describe a procedure by which we construct from graph H a graph H_m which has (2) as its degree sequence. Denote graph H as H_k where $k = 0$.

Step 1. If vertex S of H_k is adjacent to all of $T_1, T_2, \dots, T_s$ then remove vertex S and the edges incident with it to produce a graph $H_{k+1} = H_m$ with degree sequence (2).

Theorem 1.1.2 (continued 3)

Proof (continued).
Step 2. If, on the other hand, for some $1 \leq i \leq s$ vertex S is not adjacent to vertex T_i, then we modify H_k as follows. Since $d(S) = s$, then vertex S is adjacent to some vertex D_j. Since the sequence is arranged in descending order, $t_i \geq d_j$. First, if $t_i = d_j$, just exchange T_i and D_j creating a new graph H_{k+1} (and the degree sequence remains unchanged in the new graph H_{k+1}). Second, if $t_i > d_j$, then T_i has more neighbors the D_j, so there is a vertex W such that T_i is adjacent to W and D_j is not adjacent to W. In this case, remove edges SD_j and T_iW and add edges ST_i and D_jW to obtain the graph H_{k+1} which also has degree sequence (1):



Theorem 1.1.2 (continued 4)

Theorem 1.1.2. (Havel, Hakimi) Consider the following two sequences and assume sequence (1) is in descending order.
(1) $s, t_1, t_2, \dots, t_s, d_1, d_2, \dots, d_n$
(2) $t_1 - 1, t_2 - 1, \dots t_s - 1, d_1, d_2, \dots, d_n$.
The sequence (1) is graphic if and only if sequence (2) is graphic.

Proof (continued). Notice that after applying Step 2, the resulting graph H_{k+1} has one more vertex in $\{T_1, T_2, \dots, T_s\}$ to which vertex S is adjacent than does graph H_k. So we can repeatedly apply Step 1 and Step 2 producing graphs $H_1, H_2, \dots, H_{m-1}$, reducing the number of vertices in $\{T_1, T_2, \dots, T_s\}$ to which vertex S is not adjacent each time we apply Step 2. Since each H_i is a finite graph, then for some $m - 1$ we have vertex S adjacent to each of $T_1, T_2, \dots, T_s$. Finally, apply Step 1 to H_{m-1} producing graph H_m with degree sequence (2), showing that (2) is graphic, as claimed. □