Introduction to Graph Theory

Chapter 1. Basic Graph Theory

1.1. Graphs and Degrees of Vertices—Proofs of Theorems

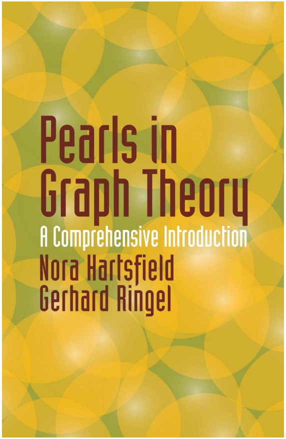

Table of contents

Theorem 1.1.1

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$$d_1 + d_2 + \cdots + d_p = \sum_{i=1}^{p} d_i = 2q.$$

Proof. By definition, an edge e of G is incident to two distinct vertices, namely its endpoints, say v_i and v_j. So any given edge e contributes (an amount of 1) to two of the degrees, say d_i and d_j. Hence each edge of G accounts for an amount of 2 in the sum $d_1 + d_2 + \cdots + d_p$. That is, the sum is twice the number of edges, $d_1 + d_2 + \cdots + d_p = 2q$, as claimed. □

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Theorem 1.1.2

Theorem 1.1.2. (Havel, Hakimi) Consider the following two sequences and assume sequence (1) is in descending order.
(1) $s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_n$
(2) $t_1 - 1, t_2 - 1, \ldots t_s - 1, d_1, d_2, \ldots, d_n$.
The sequence (1) is graphic if and only if sequence (2) is graphic.

Proof. First assume that sequence (2) is graphic. Then, by definition of "graphic," there is a graph $G_2 = (V_2, E_2)$ with degree sequence (2). We construct graph G_1 from graph G_2 by adding a single vertex S and adding s edges incident to S as follows:

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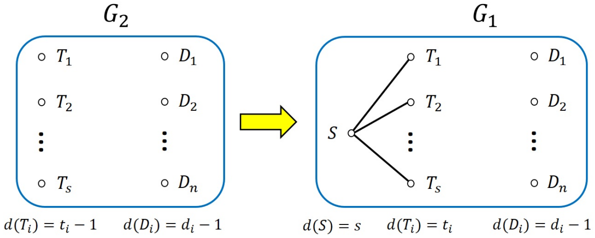

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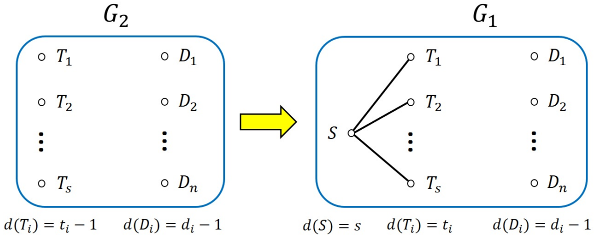

Theorem 1.1.2 (continued 1)

Proof (continued). Symbolically, construct graph $G_1 = (V_1, E_1)$ where $V_1 = V_2 \cup \{S\}$ (where S is a new vertex not in V_2) and E_1 is the set of edges consisting of all edges in E_2 along with s edges where each of these s edges has S as one endpoint and the other endpoint is one of the vertices of G_2 of degree $t_1 - 1, t_2 - 1, \ldots, t_s - 1$ (and each these s vertices of G_2 appear as an endpoint of exactly one of the new edges). In terms of the symbols introduced in the figure above,
$E_1 = E_2 \cup \{ST_i \mid i \in \{1, 2, \ldots, s\}\}$. Then in graph G_1, vertex S is of degree s, each vertex of T_i has degree t_i, and each vertex D_i has degree d_i. So graph G_1 has the sequence (1) as its degree sequence and so (1) is graphic, as claimed.

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The sequence (1) is graphic if and only if sequence (2) is graphic.

Proof (continued). Now suppose the sequence (1) is graphic. Then, by definition of "graphic," there is a graph H with degree sequence (1). Denote the vertices of H of degree t_i and T_i, the vertices of degree d_i as D_i, and the vertex of degree s as S. We describe a procedure by which we construct from graph H a graph H_m which has (2) as its degree sequence. Denote graph H as H_k where $k = 0$.

Step 1. If vertex S of H_k is adjacent to all of $T_1, T_2, \ldots, T_s$ then remove vertex S and the edges incident with it to produce a graph $H_{k+1} = H_m$ with degree sequence (2).

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Theorem 1.1.2 (continued 3)

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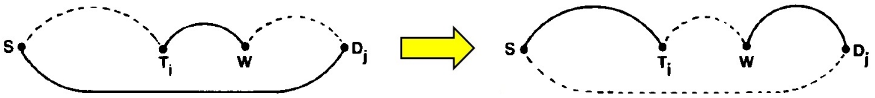
Step 2. If, on the other hand, for some $1 \leq i \leq s$ vertex S is not adjacent to vertex T_i, then we modify H_k as follows. Since $d(S) = s$, then vertex S is adjacent to some vertex D_j. Since the sequence is arranged in descending order, $t_i \geq d_j$. First, if $t_i = d_j$, just exchange T_i and D_j creating a new graph H_{k+1} (and the degree sequence remains unchanged in the new graph H_{k+1}). Second, if $t_i > d_j$, then T_i has more neighbors the D_j, so there is a vertex W such that T_i is adjacent to W and D_j is not adjacent to W. In this case, remove edges SD_j and T_iW and add edges ST_i and D_jW to obtain the graph H_{k+1} which also has degree sequence (1):



Theorem 1.1.2 (continued 4)

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The sequence (1) is graphic if and only if sequence (2) is graphic.

Proof (continued). Notice that after applying Step 2, the resulting graph H_{k+1} has one more vertex in $\{T_1, T_2, \ldots, T_s\}$ to which vertex S is adjacent than does graph H_k. So we can repeatedly apply Step 1 and Step 2 producing graphs $H_1, H_2, \ldots, H_{m-1}$, reducing the number of vertices in $\{T_1, T_2, \ldots, T_s\}$ to which vertex S is not adjacent each time we apply Step 2. Since each H_i is a finite graph, then for some $m - 1$ we have vertex S adjacent to each of $T_1, T_2, \ldots, T_s$. Finally, apply Step 1 to H_{m-1} producing graph H_m with degree sequence (2), showing that (2) is graphic, as claimed. □