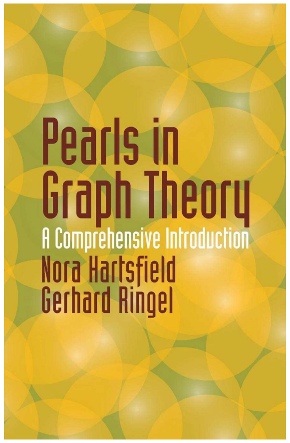
Introduction to Graph Theory

Chapter 1. Basic Graph Theory

1.3. Trees—Proofs of Theorems



Theorem 1.3.1

Theorem 1.3.1. If G is a connected graph with p vertices and q edges, then $p \leq q + 1$.

Proof. We give a proof by induction on the number of edges in G. If G has one edge then, since G is connected, it must have two vertices and the result holds. If G has two edges then, since G is connected, it must have three vertices and the result holds. So the base case is established for G having $n = 3$ (or $n = 2$) edges. Suppose the result holds for every connected graph with fewer than n edges. Let G be a connected graph with n edges and p vertices. We consider two cases.

Case 1. If G contains a cycle then we remove one edge of the cycle to create a new graph H. Then H is still connected and H has $n - 1$ edges. The number of vertices of H is the same as the number of vertices of G, namely p. By the induction hypothesis, $p \leq (n - 1) + 1$ or $p \leq n$. Then (trivially) $p \leq n + 1$ and so the number of vertices of G (namely p) is at most the number of edges of G plus 1 (namely $n + 1$).

Theorem 1.3.1 (continued)

Theorem 1.3.1. If G is a connected graph with p vertices and q edges, then $p \leq q + 1$.

Proof (continued).
Case 2. If G does not contain a cycle, then find a longest path in G. Let a and b be vertices at the end of the path. The vertex a must be of degree 1, or else G would either include a longer path (in the case that a is adjacent to a vertex not in the chosen path, contradicting the choice of the path) or G would contain a cycle (in the case that a is adjacent to another vertex of the path). Remove vertex a and the single edge incident with a to create graph H. Then H is connected and H has $p - 1$ vertices and $n - 1$ edges. By the induction hypothesis, the number of vertices of H is at most the number of edges of H plus 1; that is, $p - 1 \leq (n - 1) + 1$. So $p \leq n + 1$ and the number of vertices of G is at most the number of edges of G plus 1.

So the result now holds by Mathematical Induction. □

Theorem 1.3.2

Theorem 1.3.2. If G is a tree with p vertices and q edges, then $p = q + 1$.

Proof. We give a proof based on mathematical induction on the number of edges of G. First, if G is a tree with $q = 1$ edge then, since trees are be definition connected, G must have $p = 2$ vertices and the base case holds. Now assume that the theorem is true for all trees with fewer then n edges (the induction hypothesis).

Let G be a tree with p vertices and n edges. As in the proof of Theorem 1.3.1, select a longest path in G with a and b as the ends of the path. Then vertex a must be degree 1, or else (in the case that a is adjacent to a vertex not in the path) the path could be made longer in contradiction to the fact that it is a longest path or (in the case that a is adjacent to 2 vertices in the path) G contains a cycle in contradiction to the fact that it is a tree. Then we "subtract" vertex a from graph G together with the edge incident with a. This gives a tree H with $p - 1$ vertices and $n - 1$ edges.

Theorem 1.3.2

Theorem 1.3.2. If G is a tree with p vertices and q edges, then $p = q + 1$.

Proof. . . . tree H with $p - 1$ vertices and $n - 1$ edges. By the induction hypothesis, tree H then satisfies $(p - 1) = (n - 1) + 1 = n$. Therefore $p = n + 1$ and, since G has p vertices and n edges, the result holds tree G. Since G is an arbitrary tree with p vertices and n edges, then the claim hold for all trees with n edges.

So the result now holds by Mathematical Induction. □

Theorem 1.3.3

Theorem 1.3.3. If G is connected, and $p = q + 1$, then G is a tree.

Proof. Let graph G be connected with $p = q + 1$. ASSUME G is not a tree. Since G is connected but not a tree, then G must contain a cycle. "Subtract" an edge from G that is in the cycle and produce a graph H. Then H is still connected and H has p vertices and $q - 1$ edges. So by Theorem 1.3.1, $p \leq (q - 1) + 1$, or $p \leq q$. But we have assumed that $p = q + 1$, a CONTRADICTION. So the assumption that G is not a tree is false, and hence G is a tree as claimed. □

Theorem 1.3.A

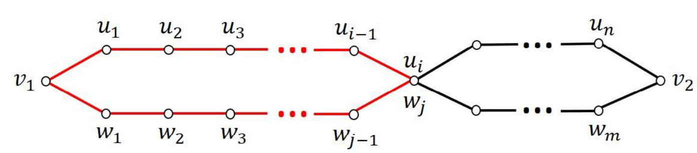
Theorem 1.3.A. Let the average degree of a connected graph G be greater than two. Then G has at least two cycles.

Proof. Let G be a connected graph, and let $d_1, d_2, \ldots, d_p$ be the degree sequence of G. Since the average degree is greater that 2, we have $2 < \dfrac{d_1 + d_2 + \cdots + d_p}{p}$. By Theorem 1.1.1, $d_1 + d_2 + \cdots + d_p = 2q$, we we must have $2 < 2q/p$ or $p < q$. Then by Theorem 1.3.2, G is not a tree. Since G is connected and not a tree, then G must contain at least one cycle C_1. "Subtract" an edge of cycle C_1 from G producing connected graph G' with $p' = p$ vertices and $q' = q - 1$ edges. Since $p < q$ then $p' = p \leq q - 1 = q'$ and by Theorem 1.3.2, G' is not a tree. Since G' is connected but not a tree, then G' contains a cycle C_2. Notice that cycles C_1 and C_2 are different since an edges of C_1 was subtracted in the creation of graph G'. Therefore G contains at least two cycles, as claimed. □

Theorem 1.3.5

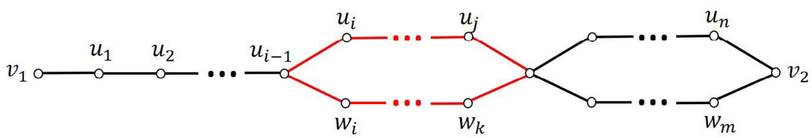
Theorem 1.3.5. A graph G is a tree if and only if there exists exactly one path between any two vertices.

Proof. First, suppose G is a tree. Let v_1 and v_2 be vertices of G. Since trees are, by definition, connected then there is a path from v_1 to v_2. ASSUME there are two (or more) paths from v_1 to v_2, say $P_1 = v_1 u_1 u_2 \cdots u_n v_2$ and $P_2 = v_1 w_1 w_2 \cdots w_m v_2$. If u_1 is distinct from w_1, then we follow P_1 until we find a vertex, $u_i = w_j$, contained in P_1 that is also in P_2 (this may be v_2). Then we have a cycle:



Theorem 1.3.5 (continued 1)

Proof (continued). If $u_1 = w_1$ then we follow path P_1 until we reach a vertex $u_i \neq w_i$ (which exists sine P_1 and P_2 are different paths joining v_1 and v_2). Then we follow path P_1 from u_{i-1} to u_i to u_{i+1}, etc. until we find a vertex in P_1 that is also in P_2 (such a vertex exists since vertex v_2 satisfies the needed condition), and the we follow path P_2 back to u_{i-1} and this gives a cycle:



In either case G contains a cycle, but this is a CONTRADICTION to the fact that G is a tree. So the assumption that there are two (or more) paths from v_1 to v_2 is false and hence there is exactly one path between v_1 and v_2. Since v_1 and v_2 are arbitrary vertices of G, then there is exactly one path between any two vertices of G.

Theorem 1.3.5 (continued 2)

Theorem 1.3.5. A graph G is a tree if and only if there exists exactly one path between any two vertices.

Proof (continued). Now suppose that G is a graph with exactly one path between any two vertices. Notice that this implies that G is connected. ASSUME that G contains a cycle $v_1 v_2 \cdots v_n v_1$. Then there are two paths from v_1 to v_n, namely the path $v_1 v_2 \cdots v_n$ and the path $v_n v_1 = v_1 v_n$. But this is a CONTRADICTION to the fact that there is exactly one path between any two vertices of G. So the assumption that G contains a cycle is false and hence G has no subgraph isomorphic to a cycle. That is, G is a tree. □

Theorem 1.3.6

Theorem 1.3.6. Every connected graph G contains a spanning tree.

Proof. If G is a tree, then the result trivially holds since G is a spanning tree of itself. If G is not a tree, then G contains a cycle. Let e_1 be an edge of the cycle and let $H_1 = G - e_1$ (that is, H_1 is the graph obtained from G by deleting edge e_1). Notice that H_1 is connected. If H_1 is a tree, then we are done. If not, then H_1 contains a cycle. Let e_2 be an edge of the cycle and let $H_2 = H_1 - e_2$. Notice that H_2 is connected. We continue inductively in this manner creating graphs $H_1, H_2, \ldots$. Since G is a finite graph, this process terminates at some graph H_i is connected and which contains no cycle; that is, H_i is a tree. So G contains the spanning tree H_i, as claimed. □