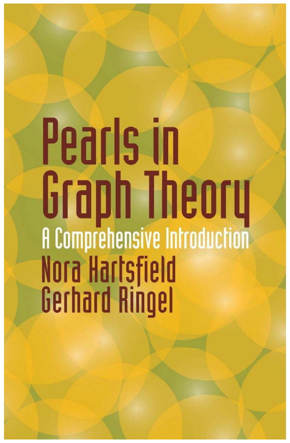
Introduction to Graph Theory

Chapter 10. Graphs on Surfaces

10.1. Rotations of Graphs—Proofs of Theorems



Theorem 10.1.1

Theorem 10.1.1. If a vertex v of a graph has degree d, then there are $(d-1)!$ different rotations of v.

Proof. We chose a first neighbor of v, a_1. There are d ways to choose a_1. We chose a second neighbor a_2 (there are $d-1$ ways to choose a_2), choose a third neighbor a_3 (there are $d-2$ ways to choose a_3), and so forth. So, by the Fundamental Counting Principle there are $d!$ orderings of the neighbors of v. However, since cyclic permutations of the ordering do not affect the rotation, we have counted each rotation d times (any of the vertices can be considered the "first" neighbor), so that the number of different rotations of v is $d!/d = (d-1)!$, as claimed. □

Theorem 10.1.2

Theorem 10.1.2. Given a connected graph with p vertices and q edges, and a rotation ρ which induces $r(\rho)$ circuits, the inequality $p - q + r(\rho) \leq 2$ holds. Furthermore, the alternating sum $p - q + r(\rho)$ is even.

Proof. We give an inductive proof on the number of cycles in the graph. For the base case, suppose there are no cycles in the graph. Then it is a tree and so is connected. Any rotation of the tree induces exactly on circuit; we justify this claim with Figure 10.1.10. Therefore $r(\rho) = 1$ for any rotation ρ.

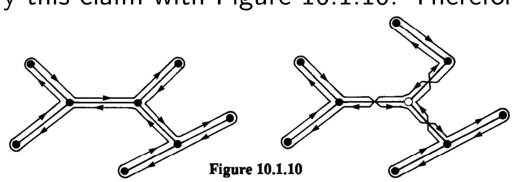


Figure 10.1.10

In a tree $q = p - 1$ by Theorem 1.3.2, so $p - q + r(\rho) = p - (p-1) + 1 = 2$ and the inequality holds, establishing the base case.

Theorem 10.1.2 (continued 1)

Proof (continued). For the induction hypothesis, suppose the claim holds for all connected graphs that have n or fewer cycles. Let G be a graph with $n+1$ cycles, and let ρ be a rotation of G. Let e be some edge on the cycle of G. Rotation ρ induces circuits in G and edge e either appears twice in the same circuit (once in each direction; see Figure 10.1.11) or it appears in two different circuits (in two different directions; see Figure 10.1.12).

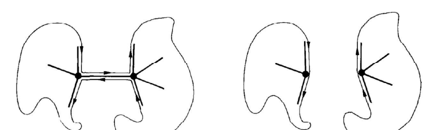


Figure 10.1.11

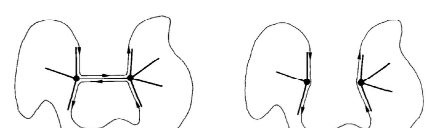


Figure 10.1.12

Theorem 10.1.2 (continued 2)

Proof (continued). Consider the graph $G - e$, and choose the rotation $\hat{\rho}$ which is the same as ρ everywhere, except at the endpoints of e where $\hat{\rho}$ is ρ with edge e deleted. If e occurs twice in one circuit of G (as in Figure 10.1.11 left), then the one circuit of G will be replaced by two circuits in $G - e$ (see Figure 10.1.11 right). Then $r(\rho) = r(\hat{\rho}) - 1$. If e occurs in two different circuits of G (as in Figure 10.1.12 left), then in $G - e$ the two circuits are replaced with one circuit (see Figure 10.1.12 right). Then $r(\rho) = r(\hat{\rho}) + 1$. So in either of these cases, $r(\rho) = r(\hat{\rho}) \pm 1$. Now graph $G - e$ has n or fewer cycles. If the original graph G has p vertices and q edges, then $G - e$ has p vertices and $q - 1$ edges. By the induction hypothesis, $p - (q - 1) + r(\hat{\rho}) \leq 2$ and $p - (q - 1) + r(\hat{\rho})$ is even. Therefore

$$p - q + r(\rho) = p - (q - 1) - 1 + r(\hat{\rho}) \pm 1 \leq p - (q - 1) + r(\hat{\rho}) \leq 2,$$

as claimed. Since $p - (q - 1) + r(\hat{\rho})$ is even and -1 ± 1 equals 0 or -2, then $p - q + r(\rho)$ is also even, as claimed. □