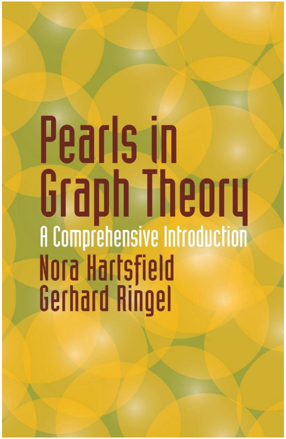
# Introduction to Graph Theory

## Chapter 10. Graphs on Surfaces

10.1. Rotations of Graphs—Proofs of Theorems



# Table of contents

# Theorem 10.1.1

**Theorem 10.1.1.** If a vertex $v$ of a graph has degree $d$, then there are $(d-1)!$ different rotations of $v$.

**Proof.** We chose a first neighbor of $v$, $a_1$. There are $d$ ways to choose $a_1$. We chose a second neighbor $a_2$ (there are $d-1$ ways to choose $a_2$), choose a third neighbor $a_3$ (there are $d-2$ ways to choose $a_3$), and so forth. So, by the Fundamental Counting Principle there are $d!$ orderings of the neighbors of $v$. However, since cyclic permutations of the ordering do not affect the rotation, we have counted each rotation $d$ times (any of the vertices can be considered the "first" neighbor), so that the number of different rotations of $v$ is $d!/d = (d-1)!$, as claimed. □

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# Theorem 10.1.2

**Theorem 10.1.2.** Given a connected graph with $p$ vertices and $q$ edges, and a rotation $\rho$ which induces $r(\rho)$ circuits, the inequality $p - q + r(\rho) \leq 2$ holds. Furthermore, the alternating sum $p - q + r(\rho)$ is even.

**Proof.** We give an inductive proof on the number of cycles in the graph. For the base case, suppose there are no cycles in the graph. Then it is a tree and so is connected. Any rotation of the tree induces exactly on circuit; we justify this claim with Figure 10.1.10. Therefore $r(\rho) = 1$ for any rotation $\rho$.

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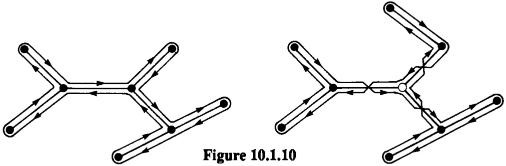

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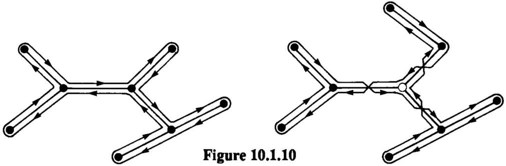

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# Theorem 10.1.2 (continued 1)

**Proof (continued).** For the induction hypothesis, suppose the claim holds for all connected graphs that have $n$ or fewer cycles. Let $G$ be a graph with $n+1$ cycles, and let $\rho$ be a rotation of $G$. Let $e$ be some edge on the cycle of $G$. Rotation $\rho$ induces circuits in $G$ and edge $e$ either appears twice in the same circuit (once in each direction; see Figure 10.1.11) or it appears in two different circuits (in two different directions; see Figure 10.1.12).

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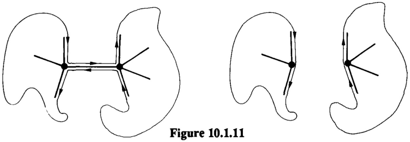
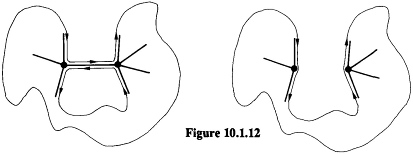


**Figure 10.1.11**



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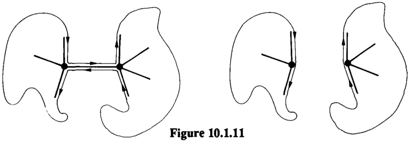

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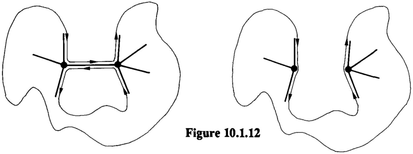

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# Theorem 10.1.2 (continued 2)

**Proof (continued).** Consider the graph $G - e$, and choose the rotation $\hat{\rho}$ which is the same as $\rho$ everywhere, except at the endpoints of $e$ where $\hat{\rho}$ is $\rho$ with edge $e$ deleted. If $e$ occurs twice in one circuit of $G$ (as in Figure 10.1.11 left), then the one circuit of $G$ will be replaced by two circuits in $G - e$ (see Figure 10.1.11 right). Then $r(\rho) = r(\hat{\rho}) - 1$.

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$$p - q + r(\rho) = p - (q - 1) - 1 + r(\hat{\rho}) \pm 1 \leq p - (q - 1) + r(\hat{\rho}) \leq 2,$$

as claimed. Since $p - (q - 1) + r(\hat{\rho})$ is even and $-1 \pm 1$ equals 0 or $-2$, then $p - q + r(\rho)$ is also even, as claimed. □

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