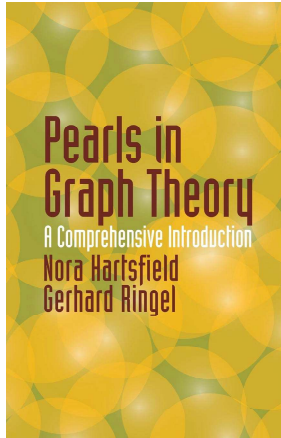


Introduction to Graph Theory

Chapter 10. Graphs on Surfaces

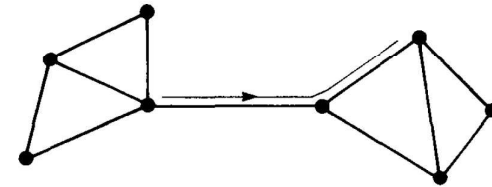
10.2. Planar Graphs Revisited—Proofs of Theorems



Theorem 10.2.1

Theorem 10.2.1. If a graph G has a bridge, then for any rotation ρ of G , the bridge occurs in both directions in one circuit induced by ρ .

Proof. The bridge must be in some circuit. Since the circuit ends are at the same vertex at which it starts, if we view the beginning vertex of the circuit containing the bridge as one end of the bridge (followed, in the circuit, by traversing the bridge) then we enter the back of the graph that does not contain the beginning vertex. Then the only way to return to the beginning vertex is to return to the bank containing this vertex and this can only be done by “crossing” the bridge again in the opposite direction:



Theorem 10.2.2

Theorem 10.2.2. If H is a subgraph of a planar graph G , then H is planar.

Proof. Let G be a planar graph with p vertices, q edges, an planar rotation ρ (so we have $p - q + r(\rho) = 2$). We consider connected graph G and graph $H = G - e$ for some edge of G . Since each subgraph of G results from successively deleting edges (and isolated vertices) from G , then if we show that $H = G - e$ is planar, this is sufficient (by induction). We consider three cases.

Case 1. Suppose one of the endpoints of e , say x , has degree 1. Then $G - e$ equals $G - x$ plus an isolated vertex (namely, vertex x), and so e is a bridge of G . Now $G - x$ has $p - 1$ vertices and $q - 1$ edges. Let $\hat{\rho}$ be the rotation of $G - x$ that is the same as ρ everywhere but at the other end of e , say vertex y , where edge e is deleted from the rotation. Now the circuit in G that contains edge e , must contain it in both directions by Theorem 10.2.1.

Theorem 10.2.2 (continued 1)

Theorem 10.2.2. If H is a subgraph of a planar graph G , then H is planar.

Proof (continued). So this circuit in G has a corresponding circuit in $H = G - x$. Any other circuit in G , is also a circuit in $H = G - x$, so that $r(\rho) = r(\hat{\rho})$ (in $G - x$ and G , respectively). So for graph $G - v$, $(p - 1) - (q - 1) + r(\hat{\rho}) = p - q + r(\rho) = 2$, so that $G - v$ is planar and, hence $G - e$ is planar.

Case 2. Suppose edge e is a bridge of G . Denote the number of edges, vertices, and circuits on one bank as p_1 , q_1 , and $r_1(\hat{\rho})$, and on the other bank as p_2 , q_2 , and $r_2(\hat{\rho})$ (where $\hat{\rho}$ is the rotation that is the same as ρ everywhere, except at the ends of edge e , where edge e is deleted from the rotation). Then we have $p_1 + p_2 = p$, $q_1 + q_2 = q - 1$, and $r_1(\hat{\rho}) + r_2(\hat{\rho}) = r(\rho) + 1$ (by Theorem 10.2.1, edge e is in one circuit of G in both directions, so this circuit breaks into two circuits in $G - e$ with one circuit in each component of $G - e$).

Theorem 10.2.2 (continued 2)

Theorem 10.2.2. If H is a subgraph of a planar graph G , then H is planar.

Proof (continued). Since G is planar then $p - q + r(\rho) = 2$, so that $p - (q - 1) + (r(\rho) + 1) = 4$, $(p_1 + p_2) - (q_1 + q_2) + (r_1(\hat{\rho}) + r_2(\hat{\rho})) = 4$, and so $(p_1 - q_1 + r_1(\hat{\rho})) + (p_2 - q_2 + r_2(\hat{\rho})) = 4$. By Theorem 10.1.2, $p_1 - q_1 + r_1(\hat{\rho}) \leq 2$ and $p_2 - q_2 + r_2(\hat{\rho}) \leq 2$, so we must have $p_1 - q_1 + r_1(\hat{\rho}) = 2$ and $p_2 - q_2 + r_2(\hat{\rho}) = 2$. So both banks of $G - e$ are planar, and hence $G - e$ is planar itself, as claimed.

Case 3. Suppose e is not a bridge. The graph $G - e$ has p vertices and $q - 1$ edges. Let $\hat{\rho}$ be the rotation of $G - e$ that is the same as ρ everywhere except at the ends of e , where edge e is deleted from the rotation. As in the proof of Theorem 10.1.2 and Figures 10.1.11 and 10.1.12 (below), we have that in $G - e$ either one circuit is replaced with two (Figure 10.1.11) or two circuits are replaced with one (Figure 10.1.12).

Theorem 10.2.2 (continued 2)

Theorem 10.2.2. If H is a subgraph of a planar graph G , then H is planar.

Proof (continued).

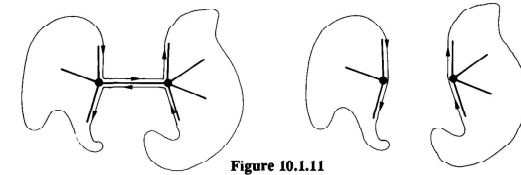


Figure 10.1.11

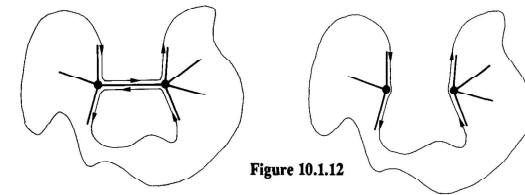


Figure 10.1.12

Theorem 10.2.2 (continued 3)

Theorem 10.2.2. If H is a subgraph of a planar graph G , then H is planar.

Proof (continued). Hence, either $r(\hat{\rho}) = r(\rho) - 1$ or $r(\hat{\rho}) = r(\rho) + 1$. If $r(\hat{\rho}) = r(\rho) - 1$, then $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) - 1 = 2$ and hence $G - e$ is planar. ASSUME $r(\hat{\rho}) = r(\rho) + 1$. Then $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) + 1 = 4 > 2$, a CONTRADICTION to Theorem 10.1.2, since $G - e$ is connected. So we must have $r(\hat{\rho}) = r(\rho) - 1$, in which case $p - (q - 1) + r(\hat{\rho}) = p - q + 1 + r(\rho) - 1 = 2$ and hence $G - e$ is planar.

In all three cases, $G - e$ is planar and, as mentioned above, every subgraph of G is planar, as claimed. \square

Theorem 10.2.3

Theorem 10.2.3. The complete bipartite graph $K_{3,3}$ is not planar.

Proof. ASSUME $K_{3,3}$ is planar. Then there is a rotation ρ of $K_{3,3}$ such that $p - q + r(\rho) = 2$. Since $p = 6$ and $q = 9$ for $K_{3,3}$, then we must have $r(\rho) = 5$. Since every edge is used twice in the 5 circuits induced by rotation ρ , then the average length of a circuit is $18/5 = 33/5$. So at least one of the circuits has length 3. But the shortest circuit in $K_{3,3}$ has length 4 (notice that no circuit is length 2 since each vertex of $K_{3,3}$ is degree 3 > 1 , and can have no circuit of length 3 since $K_{3,3}$ is bipartite), a CONTRADICTION. So the assumption that $K_{3,3}$ is planar is false and, in fact, $K_{3,3}$ is not planar, as claimed. \square

Theorem 10.2.5

Theorem 10.2.5. A maximal planar graph G with three or more vertices is connected and has no bridge.

Proof. Let G be a maximal planar graph with p vertices and q edges, and planar rotation ρ . ASSUME G is not connected. Then each of its components is planar since G is planar (by the definition of ρ). Suppose one component has p_1 vertices, q_1 edges, and $r_1(\rho)$ circuits, and another component has p_2 vertices, q_2 edges, and $r_2(\rho)$ circuits. Then $p_1 - q_1 + r_1(\rho) = 2$ and $p_2 - q_2 + r_2(\rho) = 2$, by the definition of planar of a non-connected graph. Add a new edge to G which joins the first component to the second component, creating a new component. This new edge will be a bridge in the new graph and two circuits of G will be replaced by one in the new graph, as in Case 2 of the proof of Theorem 10.2.2 (where a bridge was removed instead of introduced). So the new component has $p_1 + p_2$ vertices, $q_1 + q_2 + 1$ edges (because of the new edge), and $r_1(\rho) + r_2(\rho) - 1$ circuits (since two circuits were replaced with one).

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Theorem 10.2.5 (continued 1)

Proof (continued). In the new component we have

$$\begin{aligned} & (p_1 + p_2) - (q_1 + q_2 + 1) + (r_1(\rho) + r_2(\rho) - 1) \\ &= (p_1 - q_1 + r_1(\rho)) + (p_2 - q_2 + r_2(\rho)) - 2 = 2 + 2 - 2 = 2, \end{aligned}$$

so the new component is planar and hence the new graph with one more edge than graph G is planar. But this is a CONTRADICTION to the hypothesis that G is maximal planar. So the assumption that G is not connected is false, and hence G is connected, as claimed.

We now show that G cannot have a bridge, by contradiction. ASSUME G has a bridge. Since G has at least three vertices, one bank has more than one vertex. Denote the endpoints of the bridge by 1 and 2, and WLOG assume that vertex 3 is adjacent to vertex 1 and is on the same circuit as the bridge in a planar rotation of G (notice that the bridge is in only one circuit, by Theorem 10.2.1). See Figure 10.2.1 left (below).

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Theorem 10.2.5 (continued 2)

Proof (continued).

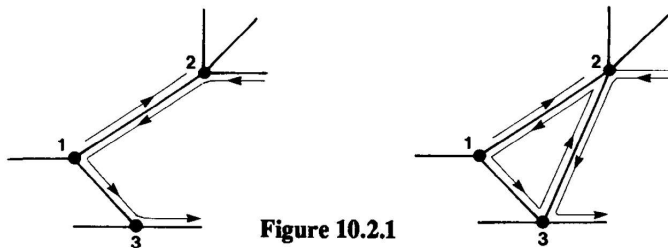


Figure 10.2.1

We now add an edge e joining vertices 2 and 3. Choose the rotation $\hat{\rho}$ of $G + e$ that is the same as ρ everywhere, except at vertices 2 and 3. If, in terms of a scheme representation of ρ at vertices 2 and 3, we have

$$2. \cdots x 1 y \cdots \quad \text{and} \quad 3. \cdots z 1 w \cdots,$$

then define $\hat{\rho}$ at vertices 2 and 3 as

$$2. \cdots x 3 1 y \cdots \quad \text{and} \quad 3. \cdots z 1 2 w \cdots.$$

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Theorem 10.2.5 (continued 3)

Theorem 10.2.5. A maximal planar graph G with three or more vertices is connected and has no bridge.

Proof (continued). We then have that $r(\hat{\rho}) = r(\rho) + 1$ (see Figure 10.2.1 right), since the one circuit containing the bridge in G becomes two circuits in $G - e$ (one of which is of length three). Graph $G + e$ has $q + 1$ edges, so $p - (q + 1) + r(\hat{\rho}) = p - q - 1 + r(\rho) + 1 = p - q + r(\rho) = 2$. But we have added an edge between two nonadjacent vertices in G , producing $G + e$ that is still planar. This CONTRADICTS the assumption that G is a maximal planar graph. So the assumption that G has a bridge is false, and hence G contains no bridge, as claimed.

We have shown that G is connected and G has no bridge, as claimed. \square

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Theorem 10.2.6

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

“Proof.” We give a proof by contradiction. ASSUME G is a maximal planar graph with planar rotation ρ and with the property that at least one circuit induced by ρ has length greater than three. Let G have p vertices and q edges. We consider three cases.

Case 1. ASSUME there is a circuit of length five or more induced by rotation ρ with all vertices distinct (and, therefore, a cycle in G), as in Figure 10.2.2 (left) below. Then at least one pair of vertices on the cycle is not adjacent, since otherwise the given graph would contain a subgraph isomorphic to K_5 , K_5 is not planar by Theorem 10.2.4, and then we would have that G is not planar by the contrapositive of Theorem 10.2.2.

Theorem 10.2.6 (continued 1)

“Proof” (continued).

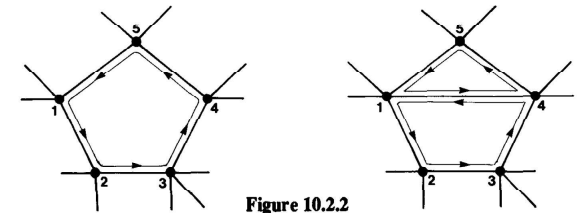


Figure 10.2.2

Suppose that vertex 1 is not adjacent to vertex 4. Add the edge e between vertices 1 and 4, obtaining a new graph $G + e$, and define the rotation $\hat{\rho}$ of $G + e$ to be the same as ρ everywhere, except at vertices 1 and 4. At vertices 1 and 4 the rotation is as suggested by Figure 10.2.2 right. Then $G + e$ has $q + 1$ edges and $r(\hat{\rho}) = r(\rho) + 1$, so that $p - (q + 1) + r(\hat{\rho}) = p - q - 1 + r(\rho) + 1 = p - q + r(\rho) = 2$, and $G + e$ is planar. But this is a CONTRADICTION, since G is a *maximal* planar graph. So the assumption that G has a circuit of length five or more with all vertices distinct is false.

Theorem 10.2.6 (continued 2)

“Proof” (continued). **Case 2.** ASSUME there is a circuit induced by ρ of length four that is a cycle. If any two of the vertices of the cycle are not adjacent, then we add an edge as in Case 1 and get a contradiction. So suppose that all four vertices are mutually adjacent, as in Figure 10.2.3 left (notice that vertices 1 and 3 are adjacent, and vertices 2 and 4 are adjacent, though the drawing given here has a crossing).

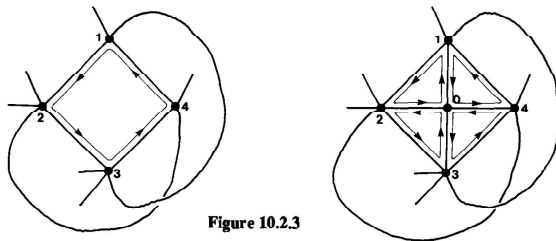


Figure 10.2.3

We construct a new graph \hat{G} from G by adding a vertex 0 and connecting it to all four vertices of the cycle (Figure 10.2.3 right).

Theorem 10.2.6 (continued 3)

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

“Proof” (continued). Define the rotation $\hat{\rho}$ of \hat{G} to be the same as ρ everywhere, except at vertices 1, 2, 3, 4, and 0. At these vertices, define $\hat{\rho}$ as suggested by Figure 10.2.3 right. So \hat{G} has $p + 1$ vertices, $q + 4$ edges, and $r(\hat{\rho}) = r(\rho) + 3$. Thus, for \hat{G} we have $(p + 1) - (q + 4) + r(\hat{\rho}) = p - 3 - 3 + (r(\rho) + 3) = p - q + r(\rho) = 2$, since G is planar. Therefore \hat{G} is planar. But \hat{G} contains a subgraph isomorphic to K_5 on vertices 0, 1, 2, 3, 4, 5. But K_5 is not planar by Theorem 10.2.4, so that G is not planar by the contrapositive of Theorem 10.2.2. But this is a CONTRADICTION to the fact that G is planar. So the assumption that G contains a circuit induced by ρ of length four that is a cycle is false.

Theorem 10.2.6 (continued 4)

“Proof” (continued). Case 3. ASSUME G contains a circuit of length at least eight with a repeated edge, as in Figure 10.2.4 left.

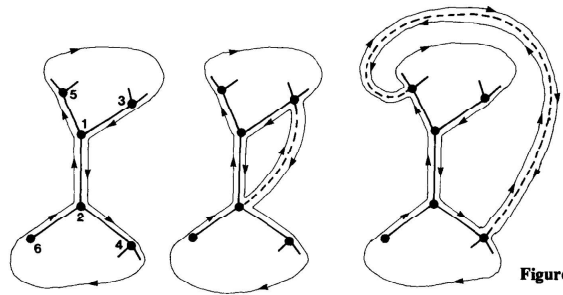


Figure 10.2.4

If any odd-numbered vertex is not adjacent to any even-numbered vertex, then add the edge joining these vertices and the new graph is still planar, giving a contradiction as in Case 1 (since this increases the number of circuits by one and increases the number of edges by one). So we must have all odd-numbered vertices adjacent to all even-numbered vertices.

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Theorem 10.2.6 (continued 5)

Theorem 10.2.6. Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.

“Proof” (continued). But then G contains a subgraph isomorphic to $K_{3,3}$. But $K_{3,3}$ is not planar by Theorem 10.2.3, so that G is not planar by the contrapositive of Theorem 10.2.2. But this is a CONTRADICTION to the fact that G is planar. So the assumption that G contains a circuit of length at least eight with a repeated edge is false.

We have shown that in G : there is no circuit induced by ρ of length four that is a cycle (in Case 2), there is no circuit of length five or more induced by rotation ρ with all vertices distinct (in Case 1), and there is no circuit of length at least eight with a repeated edge (in Case 3). From this, the text concludes that every circuit induced by ρ has length three. So... \square

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Theorem 10.2.7

Theorem 10.2.7. In a maximal planar graph with p vertices, $p \geq 3$, and q edges we have $q = 3p - 6$.

Proof. By Theorem 10.2.6, for any planar rotation ρ of a maximal planar graph, every circuit induced by ρ has length three. Since every edge is present twice in the collection of circuits, and there are $r(\rho)$ circuits (each of length three), then $3r(\rho) = 2q$. Hence $p - q + r(\rho) = p - q + 2q/3 = 2$ (because the graph is planar), which implies $q = 3p - 6$, as claimed. \square

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Theorem 10.2.9

Theorem 10.2.9. A connected graph that can be drawn in the plane with no crossings has a planar rotation. That is, graphs that are planar under the definition in Chapter 8 are also planar under the definition in Chapter 10.

Proof. We give a proof by picture. Let G be a connected graph with p vertices and q edges that can be drawn in the plane with no crossings. This results in some number of regions (or “faces”) and we assign a rotation to G by giving every vertex a clockwise rotation. See Figure 10.2.5. Then the circuits induced by ρ are the boundaries of the faces in the drawing, and hence $r(\rho) = r$. \square

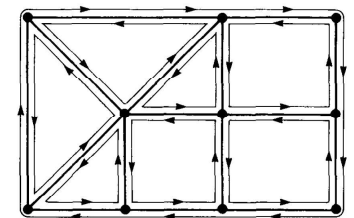


Figure 10.2.5

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